Dynamic and Streaming Algorithms for Union Volume Estimation

Sujoy Bhore Karl Bringmann Timothy Chan Yanheng Wang

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e.g., triangles, balls, polytopes

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e.g., triangles, balls, polytopes
inserted and deleted on the fly

Algorithms in the Oracle Model [Karp, Luby ’83]

algorithm

$(1 \pm \varepsilon)\,\mathrm{vol}\left(\bigcup_i X_i\right)$

oracle X_i

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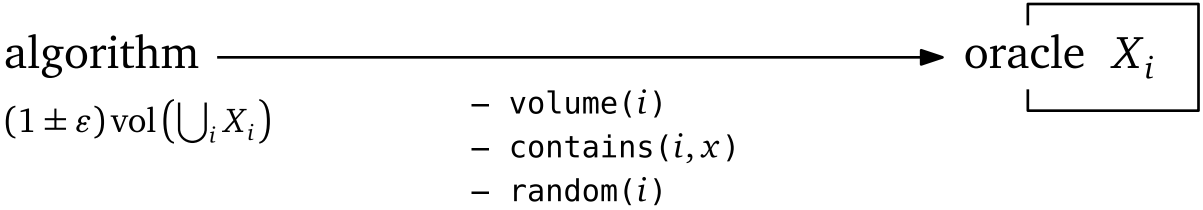

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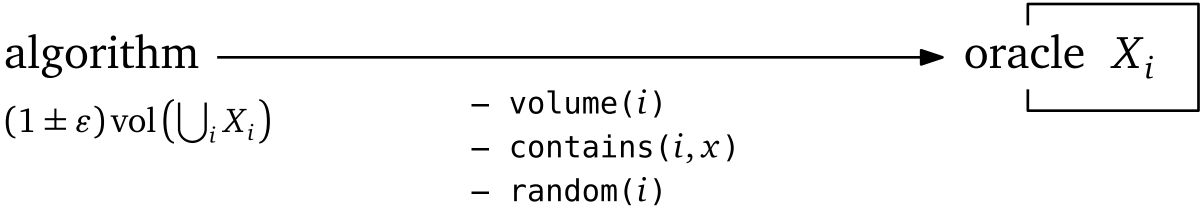

algorithm $\longrightarrow$ oracle X_i

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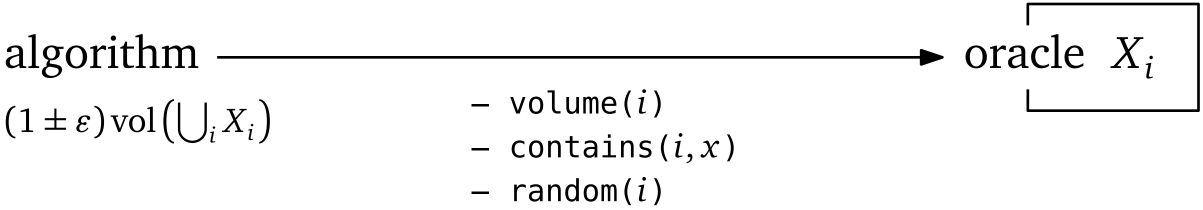

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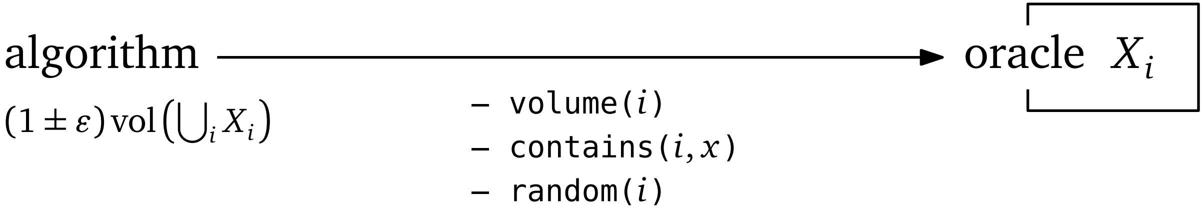

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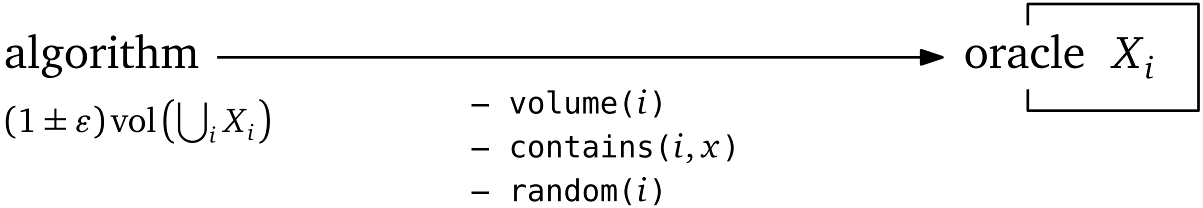

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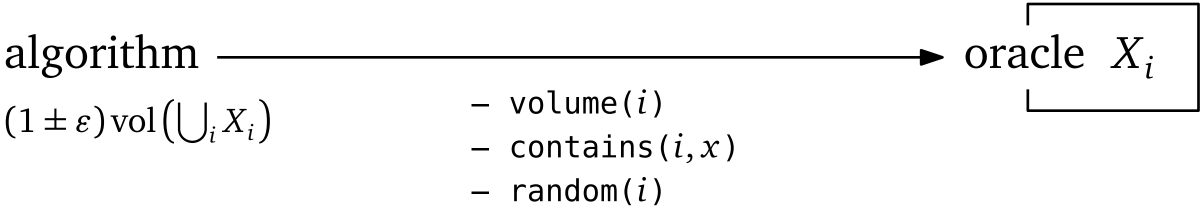

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$\widetilde{O}(1/\varepsilon^2)$ update time and $\widetilde{O}(n+1/\varepsilon^2)$ space

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Definition: ℓ-sample of a finite set U

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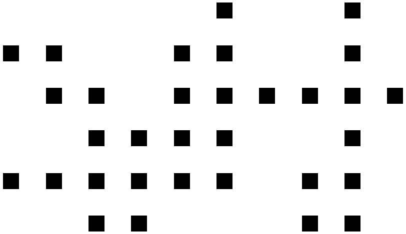

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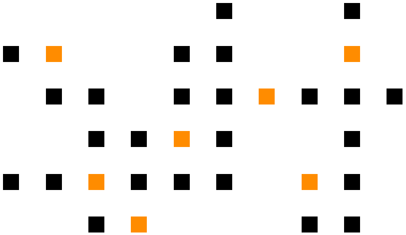

pick each point independently
with probability $2^{-\ell}$

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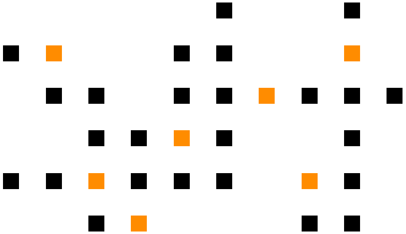

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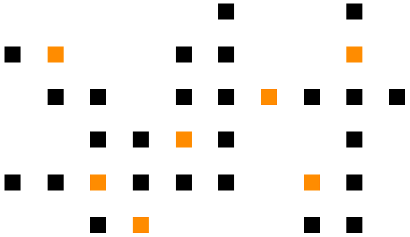

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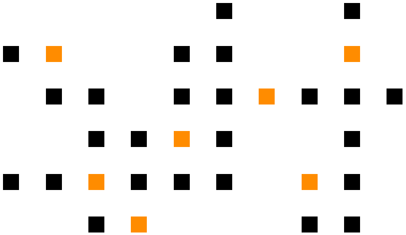

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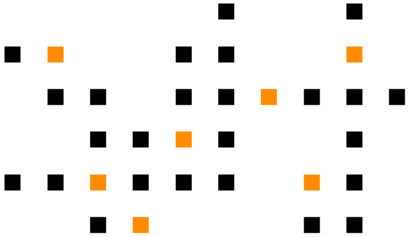

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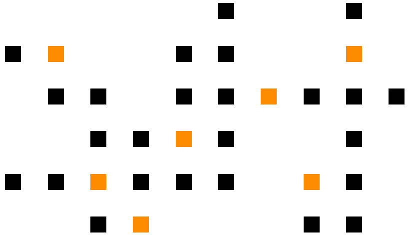

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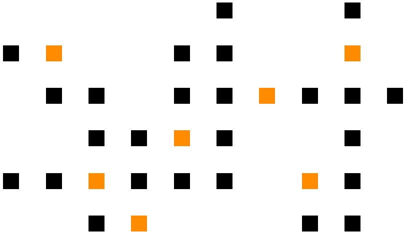

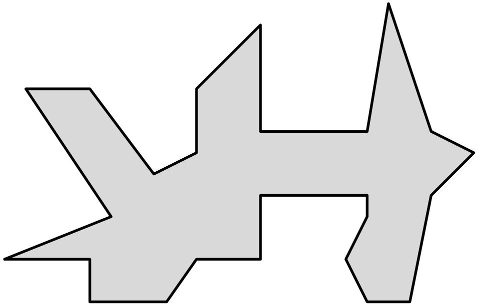

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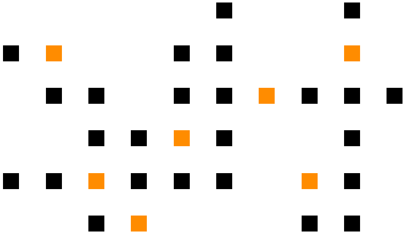

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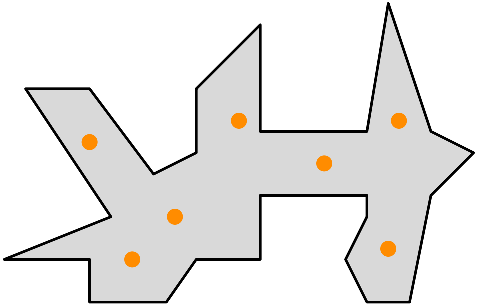

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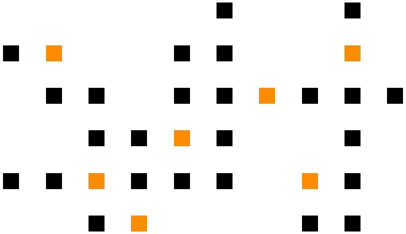

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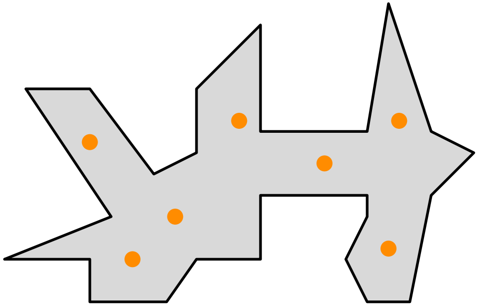

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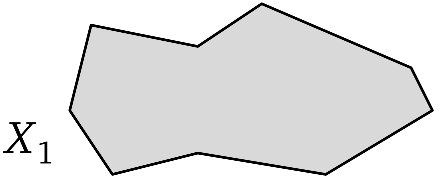

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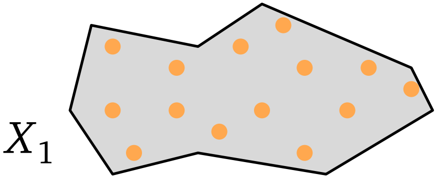

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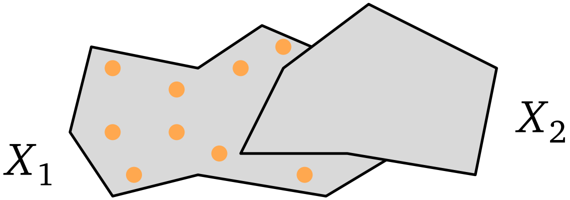

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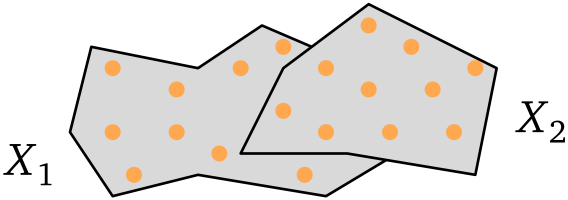

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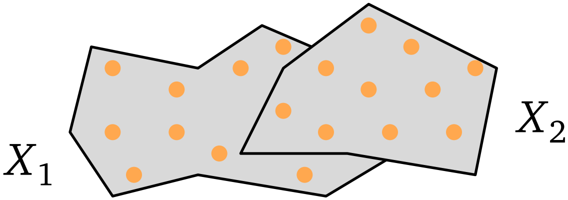

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upon insertion of X_i,

$S := S \setminus \{x : \texttt{contains}(i, x)\}$

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initially $\ell := 0$
if $|S| \gg 1/\varepsilon^2$ then
  $\ell := \ell + 1$
  subsample S

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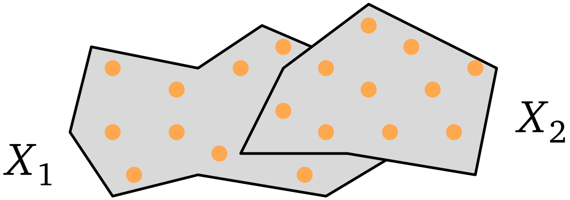
`random`(i) for N times

ℓ-sample from U:



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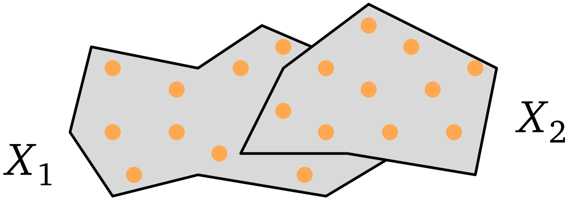

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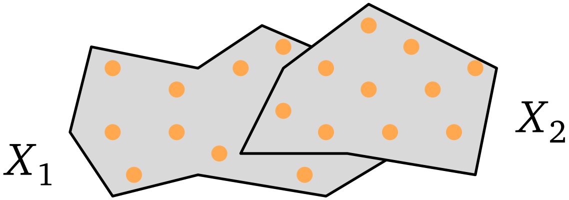

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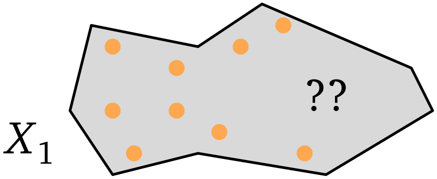

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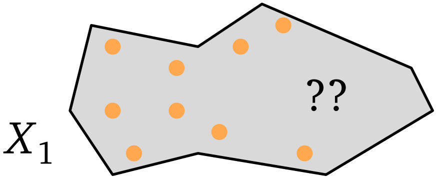

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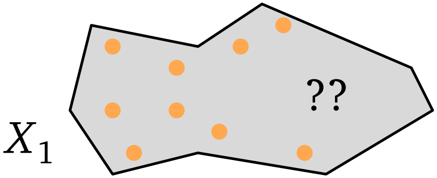

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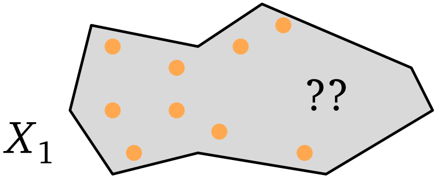

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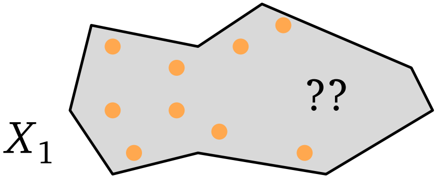

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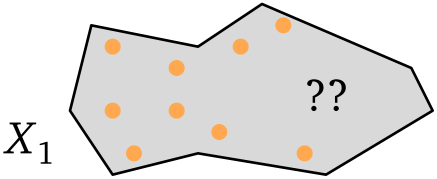

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The number of points $|S|$ drops over time
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A Decremental Algorithm

↓

A Fully Dynamic Algorithm?

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the logarithmic method [Bentley, Saxe ’80]

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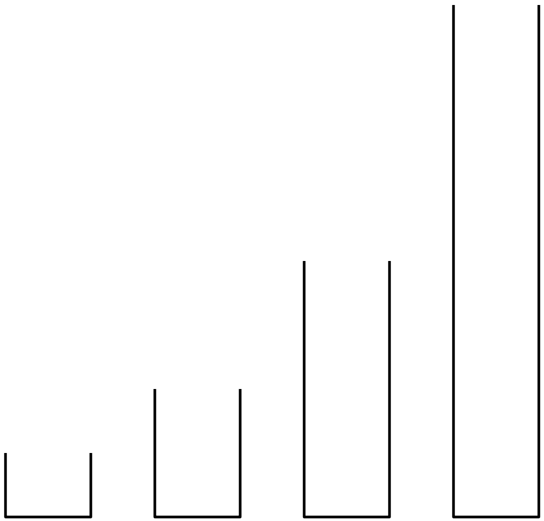


each being a decremental data structure

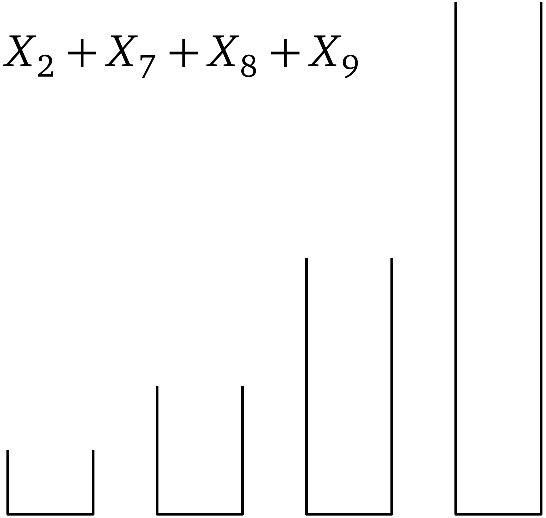
A Decremental Algorithm

↓ the logarithmic method [Bentley, Saxe ’80]

A Fully Dynamic Algorithm?

Example:

$+X_1 + X_2 + X_3 + X_4 + X_5 - X_3 + X_6 - X_2 + X_7 + X_8 + X_9$



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A Decremental Algorithm

↓ the logarithmic method [Bentley, Saxe '80]

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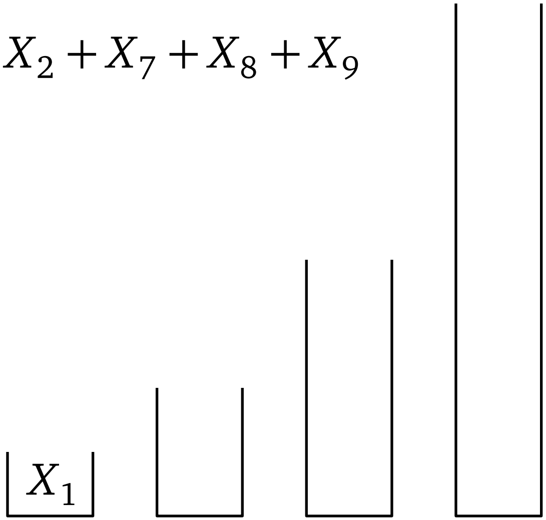

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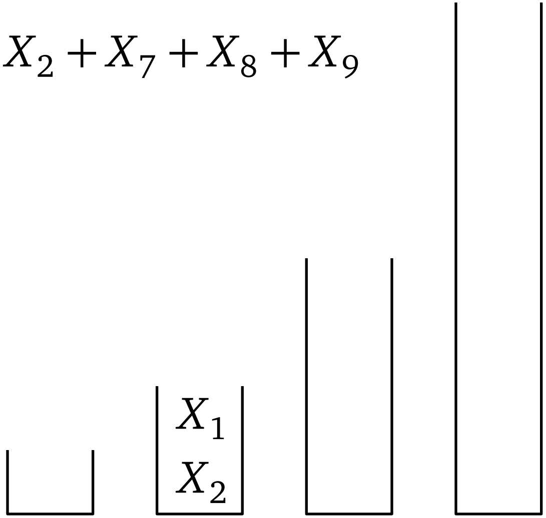

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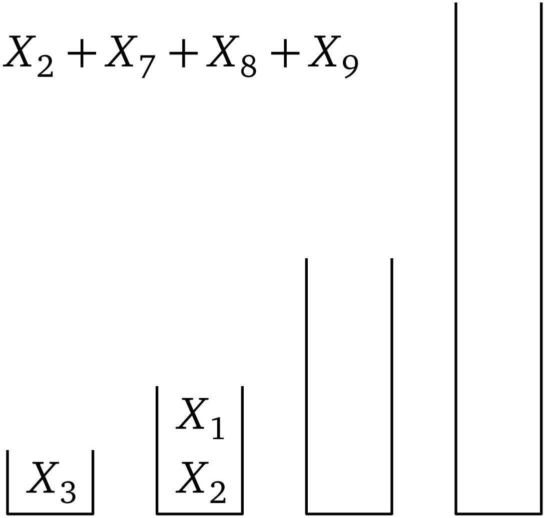

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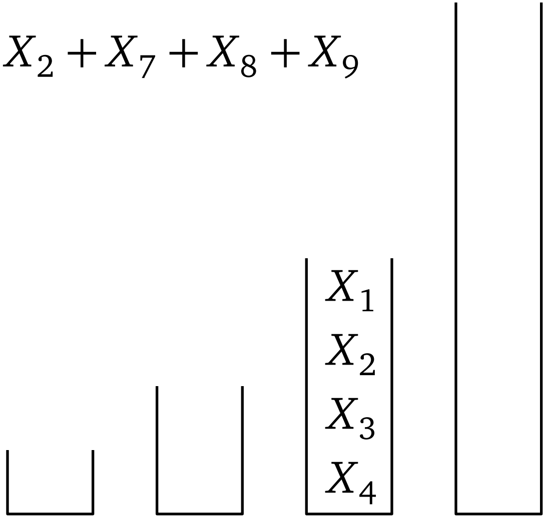

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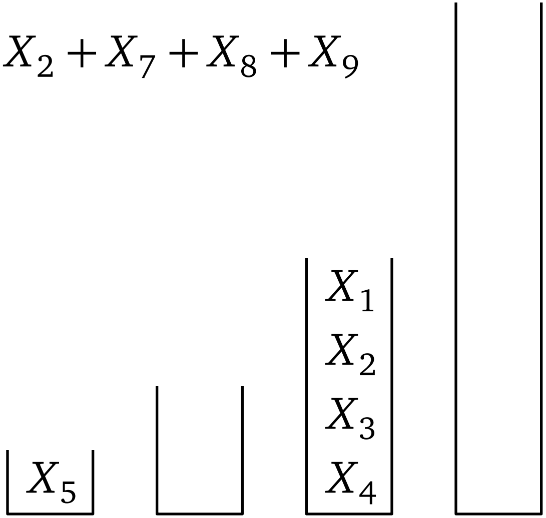

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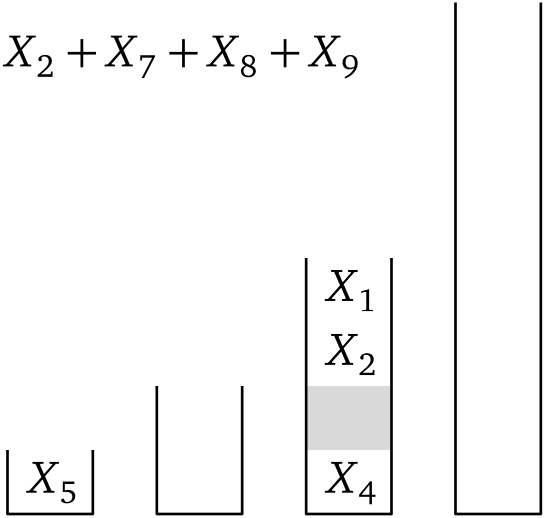

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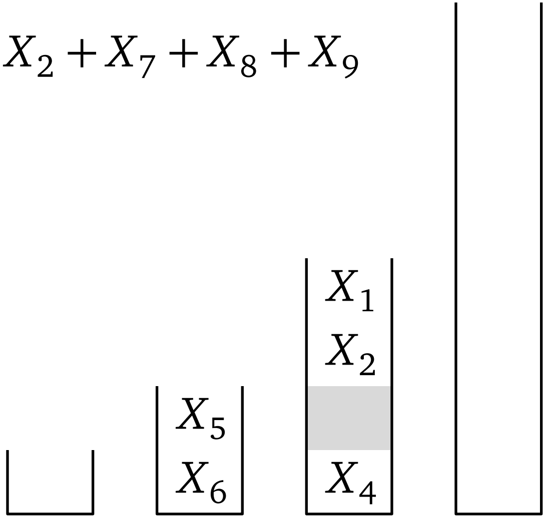

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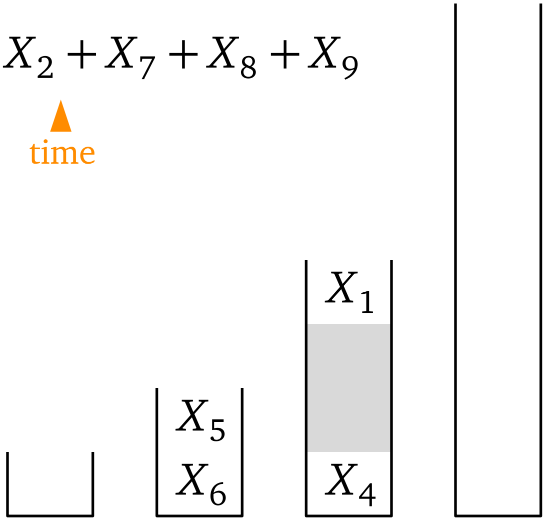

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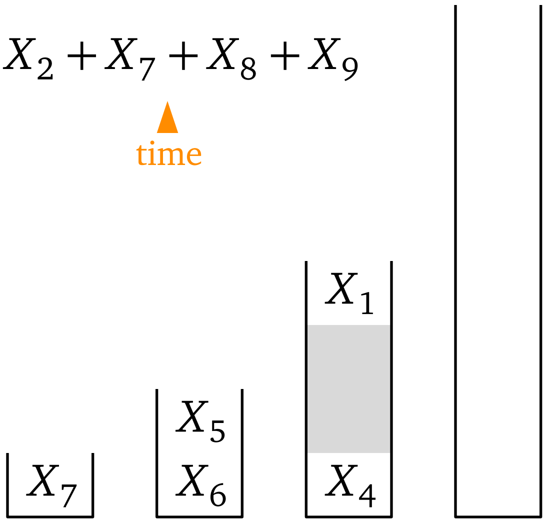

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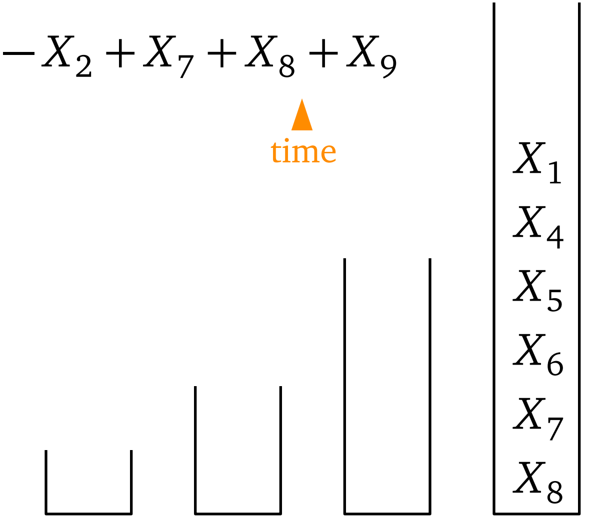

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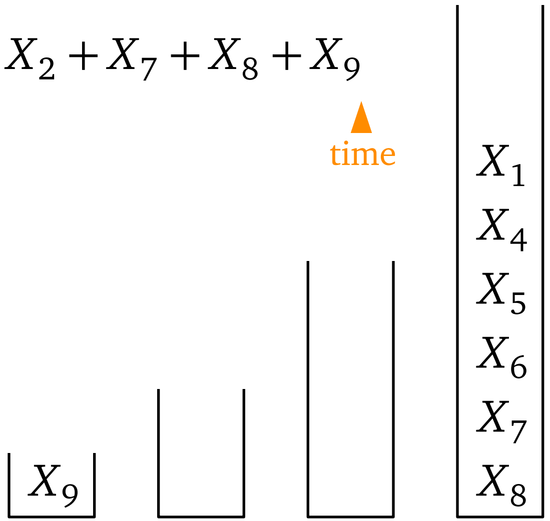

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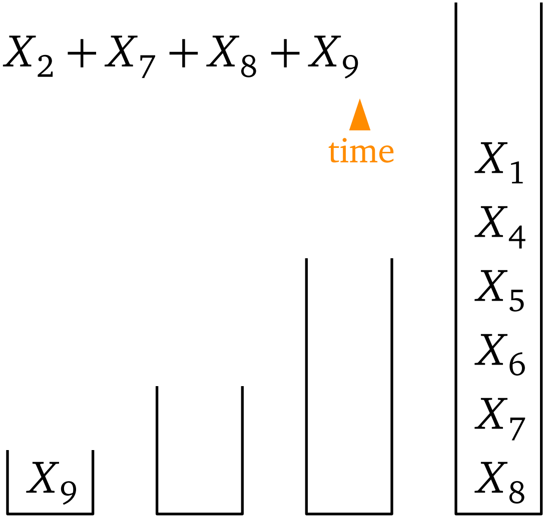
amortized overhead $=$ polylog(n)



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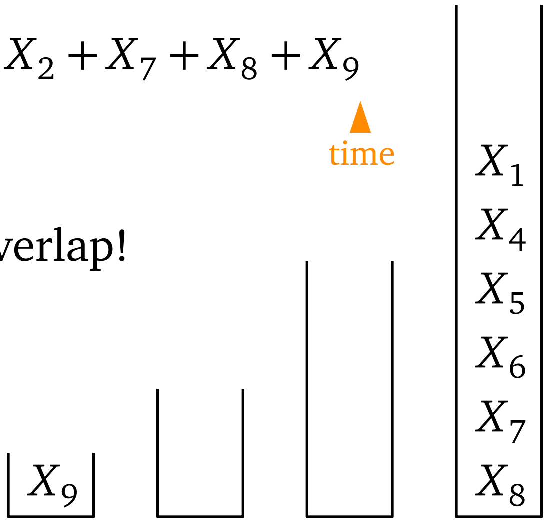

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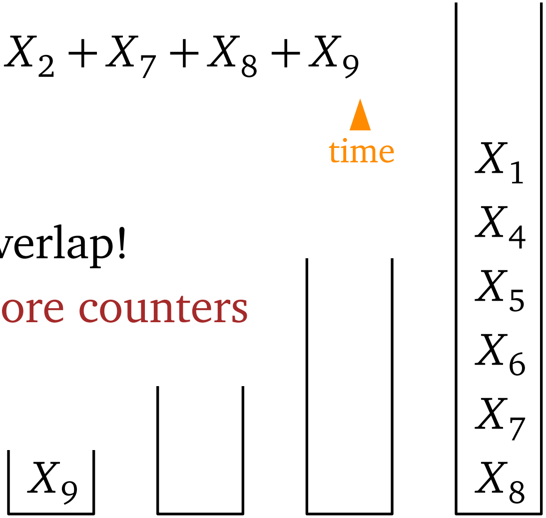

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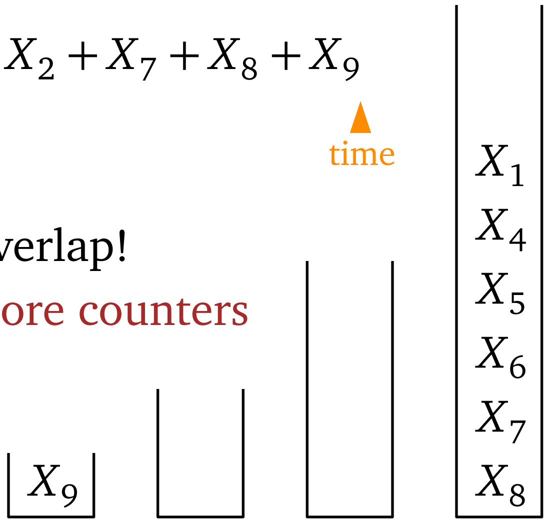
$\widetilde{O}(1/\varepsilon^2)$ amortized update time

$\widetilde{O}(n + 1/\varepsilon^2)$ space



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Improving Space Complexity

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1. **Sliding window**

   $X_1 \, X_2 \, X_3 \, X_4 \, X_5 \; \cdots$

Improving Space Complexity

1. Sliding window

$X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; \cdots$

▲

now

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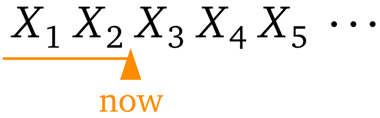

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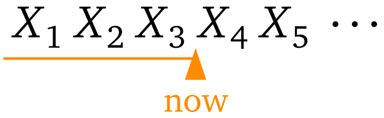

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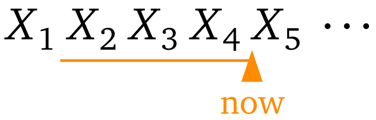

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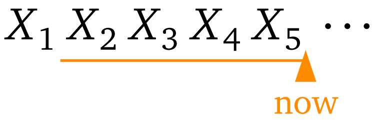

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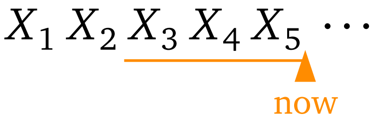

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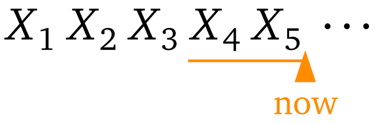

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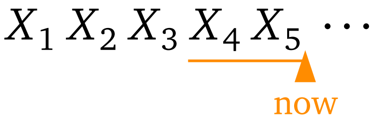

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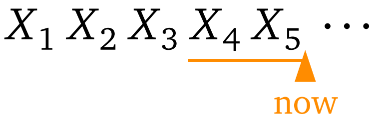

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2. Convex bodies in $\mathbb{R}^d$

If each object contains an r-ball and is contained in an R-ball, then can achieve $\mathrm{polylog}(\frac{nR}{\varepsilon r})/\varepsilon^2$ update time and space

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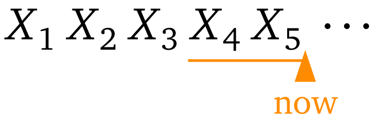

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