A Perfect Sampler for Hypergraph Independent Sets

Guoliang Qiu, Yanheng Wang, Chihao Zhang 06.07.2022

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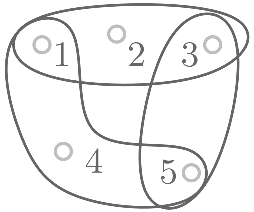

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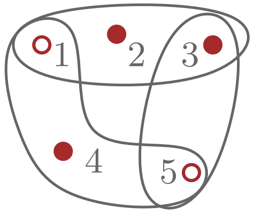

$\sigma : [n] \to \{\bullet, \circ\}$ such that no hyperedge is fully-$\bullet$

A Perfect Sampler for Hypergraph Independent Sets

outputs an independent set uniformly at random

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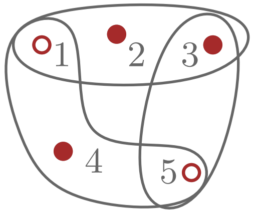

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Motivation and Results

For k-uniform d-regular hypergraphs,

- $d > 5 \cdot 2^{k/2}$: no poly-time approximate sampler assuming RP $\neq$ NP [BGGGS19]
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Result: A sampler that outputs a *perfectly* uniform independent set a.s. Its expected running time is polynomial, assuming

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- simpler analysis? ⟶ systematic scan + witness structure for analysis [HSW21]

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The Systematic Scan

Update schedule & rule:

time t

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CFTP transformation: If we can design a routine that detects coalescence, then we can turn it into a perfect sampler!

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Witness Directed Graph Up to T

Nodes. *foreach* hyperedge C:

foreach time $t = 1, \ldots, T$:

if $v_t \in C$ *then* create a node $e_{C,t}$

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$n = 4$

$C := \{1, 2, 4\}$

$D := \{2, 3\}$

t	1	2	3	4	5	6	7	$\cdots$
v_t	1	2	3	4	1	2	3	$\cdots$

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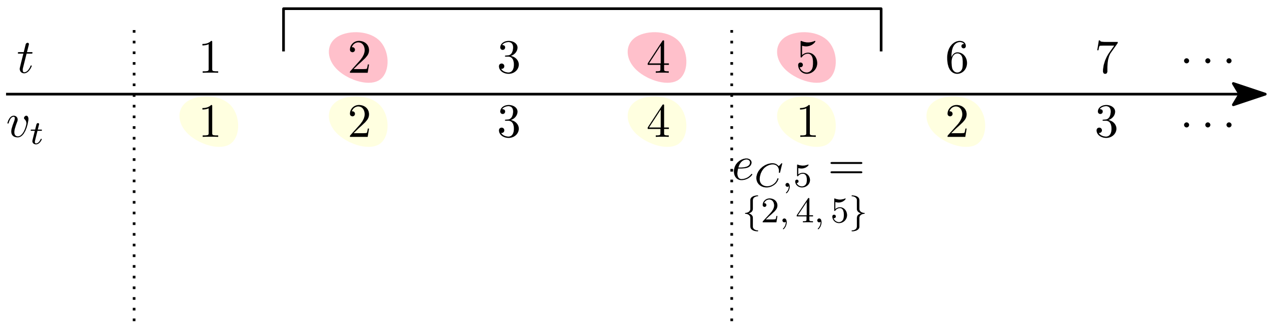

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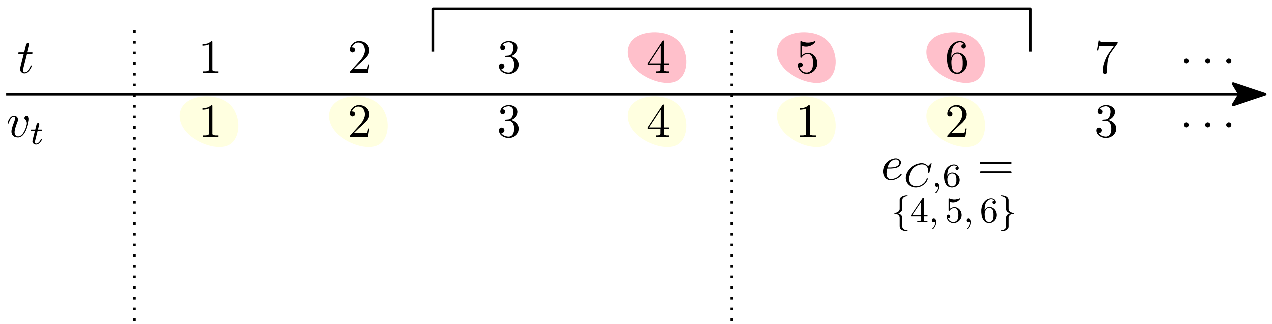

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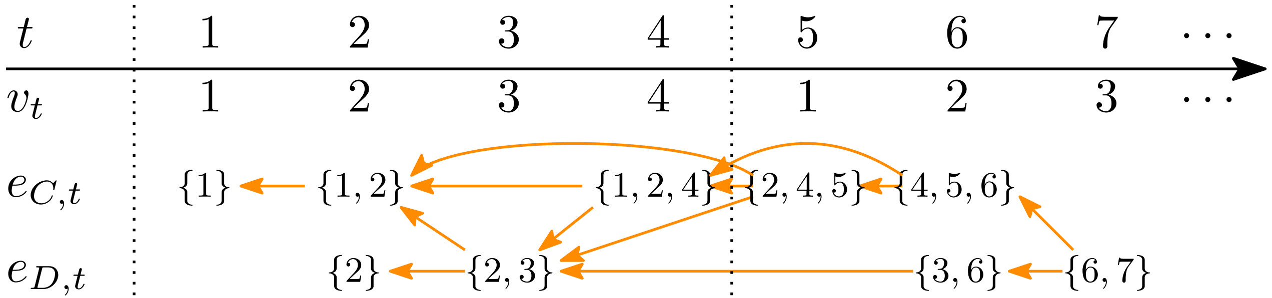

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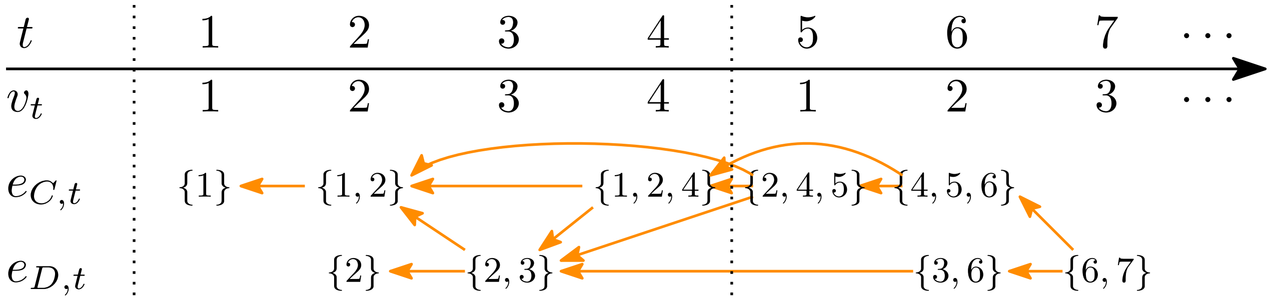

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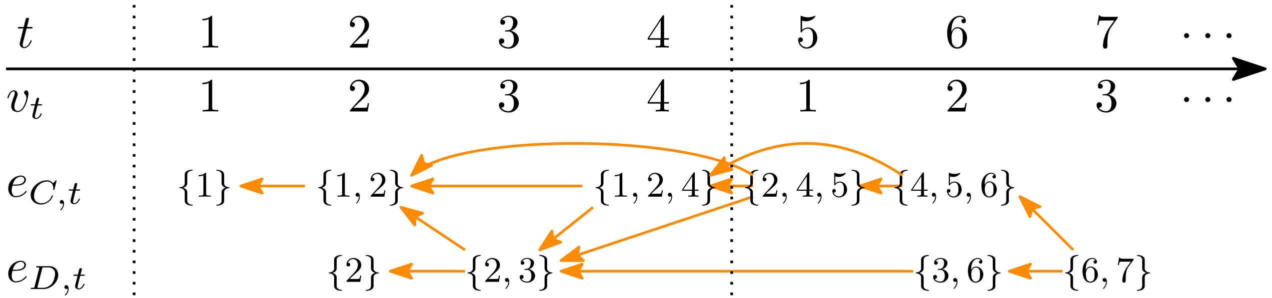

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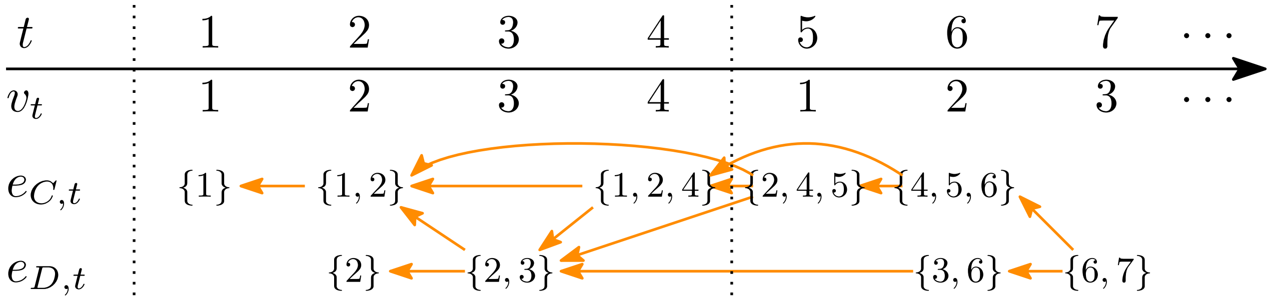

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Sketch: Map path P to a labelled tree $\mathcal{T}_P$. Then the product corresponds to the probability that $\mathcal{T}_P$ is generated by a suitable G-W process. Hence $\sum_P \text{product}(P) \leq 1$.

When Hypergraph Is k-Uniform and d-Regular

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Key: If #extensions is large, then the “intersection” between e_ℓ and Q is small

thus $\mathbb{P}[\text{(ii)} \mid \text{(i)}]$ is small