Jensen's Inequality, Partition Functions and Models with Ternary Interactions

Yanheng Wang

Prof. Dominik Scheder, *advisor*

Motivation

k-CNF $\quad \bigwedge_i \ (\ell_{i1} \vee \cdots \vee \ell_{ik})$

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a *literal*

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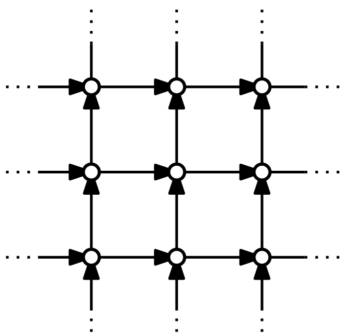

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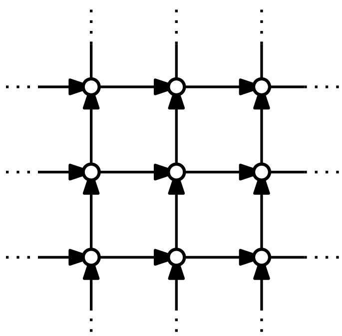

$(k-1)$-regular, n vertices

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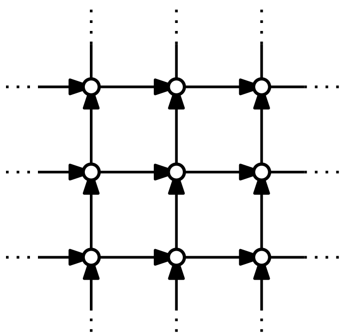

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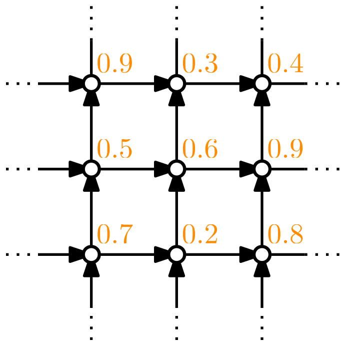

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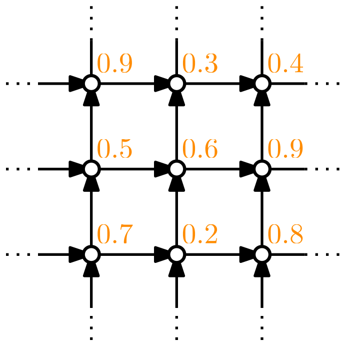

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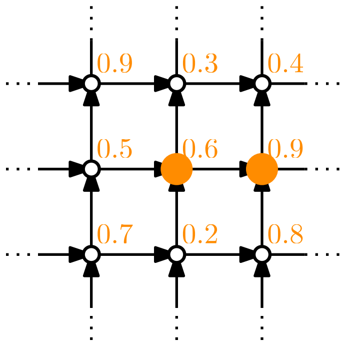

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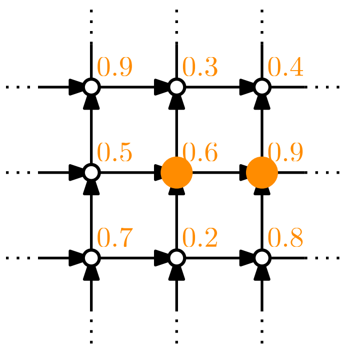

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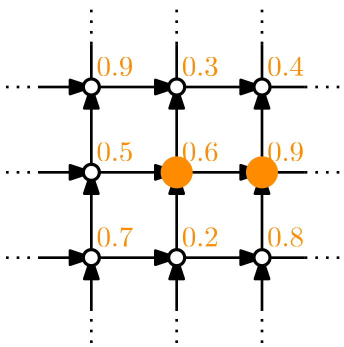

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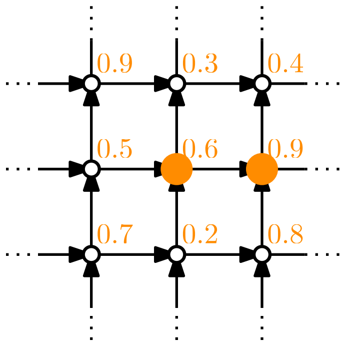

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$\mathbb{E}(2^S) \ \geq 2^{\mathbb{E}(S)} = 2^{n/k}$ $> 1.2599^n$ when $k = 3$

Main Results

For square grid, $\mathbb{E}(2^S) > 1.2711^n$. (Jensen: $\mathbb{E}(2^S) > 1.2599^n$)

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For general $(k-1)$**-regular digraphs,**

$$\mathbb{E}(2^S) \leq \left(\frac{1}{2} + \frac{[(k-2)!]^2}{(2k-3)!}(k-1)4^{k-2}\right)^{\frac{n}{k-1}} = 2^{n \cdot \Theta(\log k/k)}.$$

(Jensen: $\mathbb{E}(2^S) \geq 2^{n/k}$)

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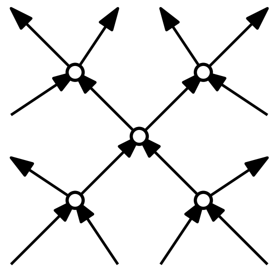

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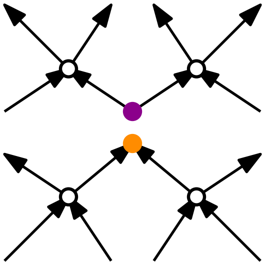

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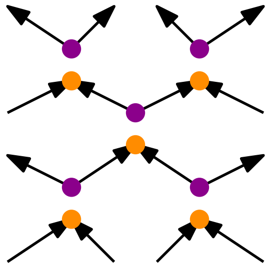
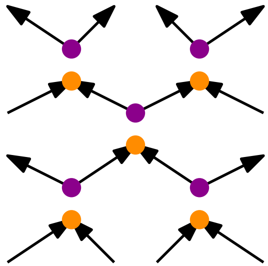


General Upper Bound

Bipartite
Model



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regular; $2n$ vertices

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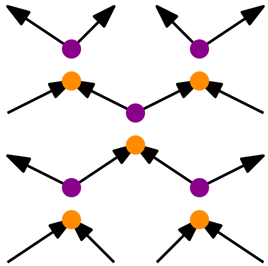

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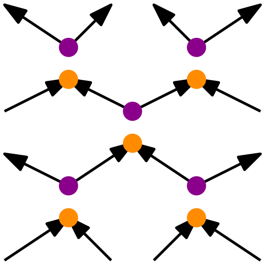

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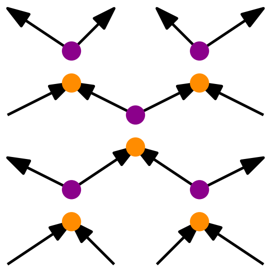

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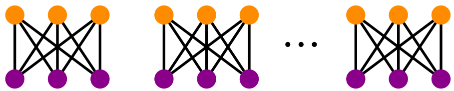
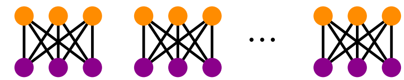
$K :=$ 

(n/k copies of complete bipartite graphs)

Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $K :=$ 

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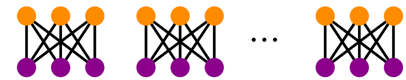
Theorem $\mathbb{E}_{G'}(2^S) \le \mathbb{E}_K(2^S)$ $\quad K :=$ 

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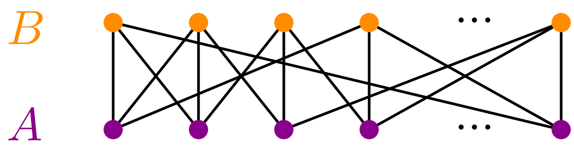
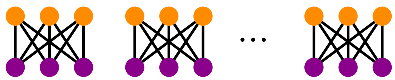


Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $K :=$ 

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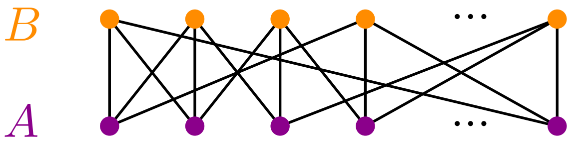
G' : 

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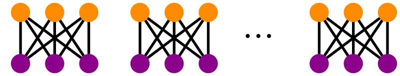
Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $\quad K :=$ 

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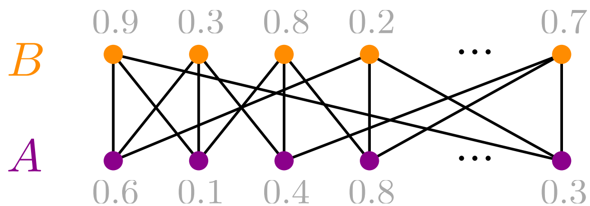

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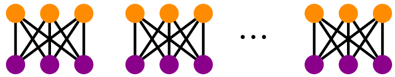
Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $K :=$ 

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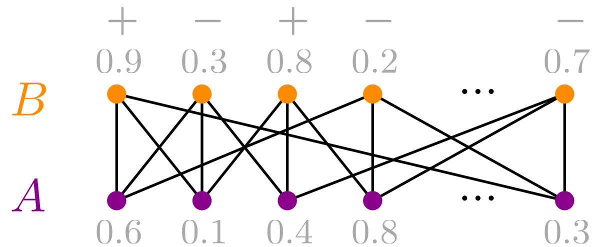

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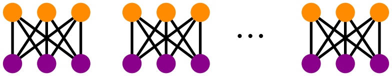

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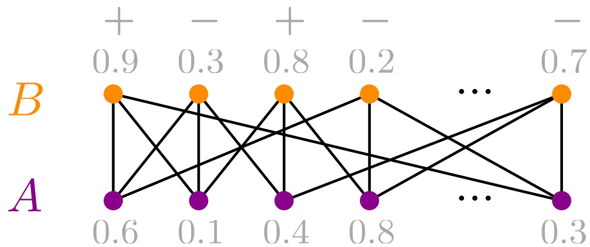

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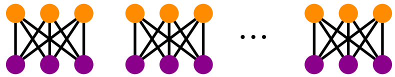

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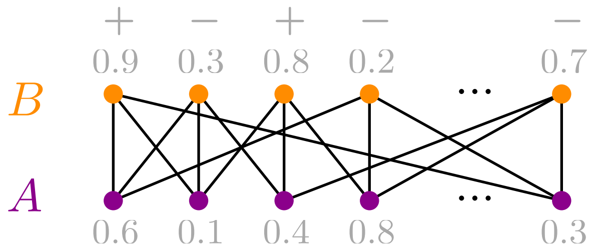

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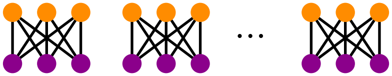

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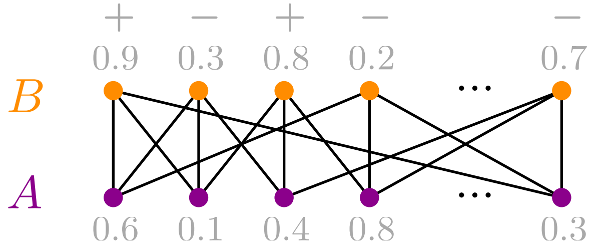

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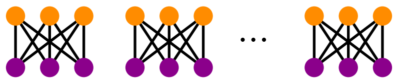
Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $K :=$ 

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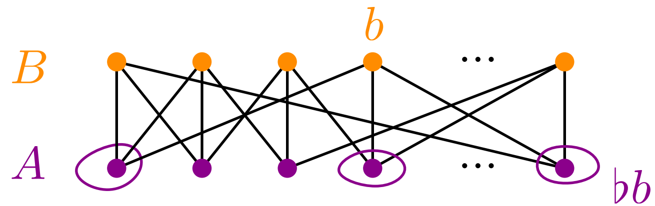
G' : 

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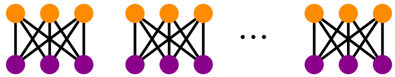
Theorem $\mathbb{E}_{G'}(2^S) \le \mathbb{E}_K(2^S)$ $K :=$ 

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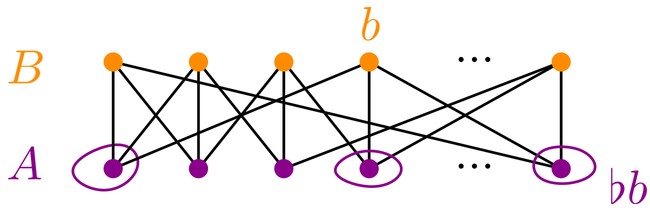
G' : 

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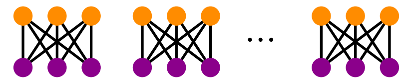
Theorem $\mathbb{E}_{G'}(2^S) \le \mathbb{E}_K(2^S)$ $K :=$ 

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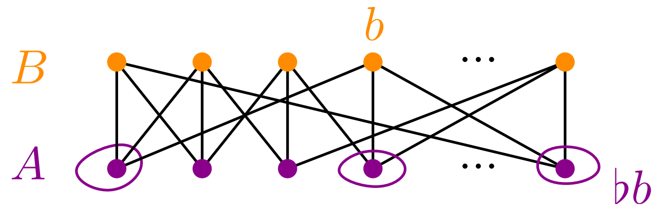
G' : 

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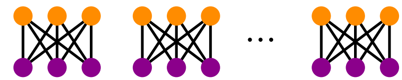
$$\le \tfrac{1}{k} \textstyle\sum_{b \in B} H(x_{\flat b}) \qquad \le \textstyle\sum_{b \in B} H(x_b, R_b \mid x_A)$$

Let $M_b := \max x_{\flat b}$ and condition on it!

Theorem $\mathbb{E}_{G'}(2^S) \le \mathbb{E}_K(2^S)$ $\quad K :=$ 

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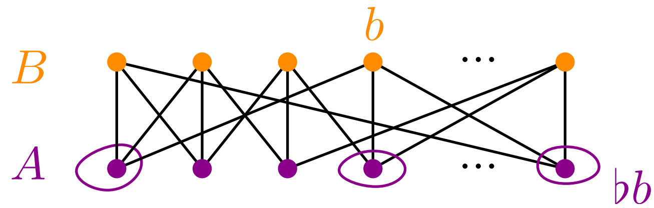

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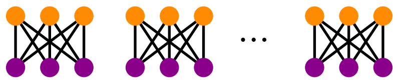
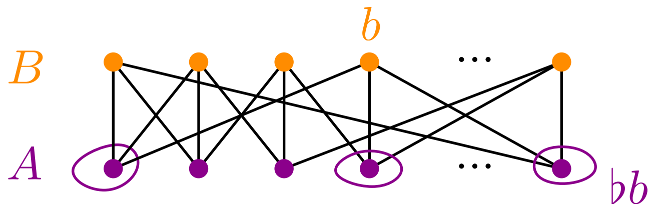
Let $M_b := \max x_{\flat b}$ and condition on it!

- If M_b is small then $H(x_{\flat b})$ is lowered
- If M_b is large then $H(x_b, R_b \mid x_A)$ is lowered

Theorem $\mathbb{E}_{G'}(2^S) \leq \mathbb{E}_K(2^S)$ $K :=$ 

Proof Sketch

G' : 

$$\Omega := \left\{ \begin{array}{ll} x \in [0,1]^{2n}, & R \in \{+,-\}^n \\ S_b(x) = 0 \implies & R_b \text{ is } - \end{array} \right\}$$

$$\log \mathbb{E}_{G'}(2^S) \quad \longleftrightarrow \quad H(x,R)$$

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(Kahn's entropy argument)

Recap

Proved $$2^{n/k} \leq \mathbb{E}(2^S) \leq 2^{n \cdot \Theta(\log k/k)}.$$

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Likely...	Structure vanishes when girth is large
Question	Asymptotics when girth is large?

Q&A Time!