DELAUNAY TRIANGULATION

"If a triangle T is skinny, then its circumcircle C_T will be relatively large, hence having a good chance of catching other points inside," observed Delaunay.

Therefore, we expect that most skinny triangles are excluded if we enforce the following property on triangulation:

<u>(Strong) Delaunay Property</u>

$\forall T \in \mathcal{T}$, the circumcircle C_T is empty, i.e. containing no point from S in its interior.

A triangulation satisfying the Delaunay property is called a Delaunay triangulation.

It is not clear yet if Delaunay triangulation really exists. Towards an existence proof (or disproof), we simplify the property a little by considering only "local constraints".

<u>Weak Delaunay Property</u>

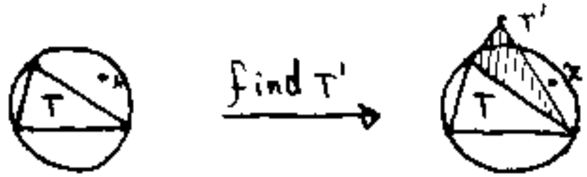
$\forall T \in \mathcal{T}$, the circumcircle C_T contains none of the vertices of adjacent triangles in its interior.

So we only have to check ≤ 3 local points for each $T \in \mathcal{T}$ in order to verify this property. On contrary, we must check all points when verifying the strong Delaunay property. But somewhat surprisingly, we don't lose anything by such weakening:

Lemma 19.
Weak Delaunay property $\Rightarrow$ strong Delaunay property.

Proof. Suppose the strong Delaunay property is violated. Then there exists a triangle

$T \in \mathcal{T}$ whose circumcircle C_T contains some $x \in S$. Among all "violators" (T, x), we choose the one pair (T, x) s.t. $\text{dist}(T, x)$ is minimised.



Let $T' \in \mathcal{T}$ be the adjacent triangle of T that lies on the same side of x. Such T' must exist, for otherwise x shall not be covered by $\mathcal{T}$.

The weak Delaunay property ensures that the other vertex of T' must be outside (or on) C_T. So obviously $C_{T'}$ encloses x. But then $\text{dist}(T', x) < \text{dist}(T, x)$, contradicting minimality. ∎

Corollary 20.

$\mathcal{T}$ is a Delaunay triangulation
$\Leftrightarrow$ $\mathcal{T}$ satisfies strong Delaunay property
$\Leftrightarrow$ $\mathcal{T}$ satisfies weak Delaunay property
$\Leftrightarrow$ $\forall$ adjacent $T, T' \in \mathcal{T}$ whose vertices are in convex position, we have: C_T doesn't contain vertices of T'.

Proof. Exercise. ∎

The corollary almost leads to a "local-fix" pro of constructing Delaunay triangulation from an arbitrary initial triangulation:

"While $\exists$ adjacent $T, T' \in \mathcal{T}$ whose vertices are in convex position but C_T contains a vertex of T', we do a local modification so that the condition no longer holds."

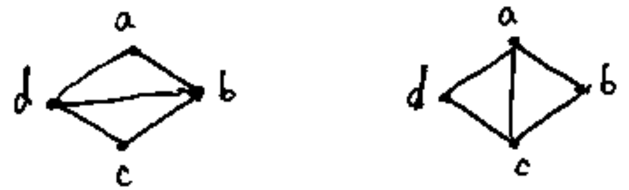
Of course, this is just a preliminary idea. What is the "local modification" we need? Does the procedure ever terminate? We have to answer these questions, but the first one is particularly easy:

Lemma 21.

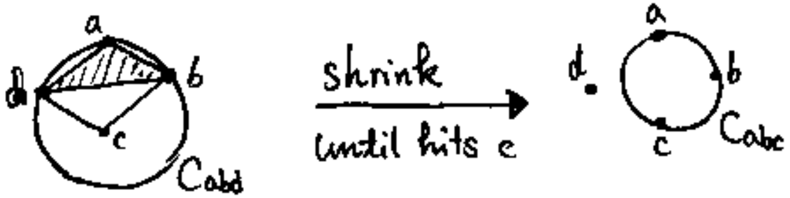
Given 4 non-collinear points in convex

position in $\mathbb{R}^2$, we could always find a Delaunay triangulation of them. Moreover, if the 4 points are non-circular, then the Delaunay triangulation is unique.

Proof. We will show that one of the pictures below is a valid Delaunay triangulation for points a, b, c, d:



If the one on the left is Delaunay, then we are done. If it is not, then we know $c \in C_{abd}$. Imagine shrinking the circle



while keeping a and b intact, until we hit c at some moment. The circle we derive is exactly C_{abc}, and obviously $d \notin C_{abc}$. Via a similar argument, we could show $b \notin C_{acd}$. Therefore, the picture on the right is exactly a Delaunay triangulation.

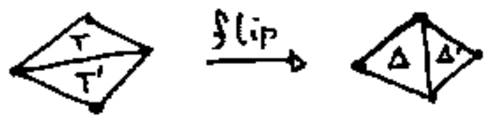
The uniqueness claim is just a by product and we leave it as an exercise. ■

With this lemma, our "local fix" procedure could be described clearly:

Algorithm LawsonFlip

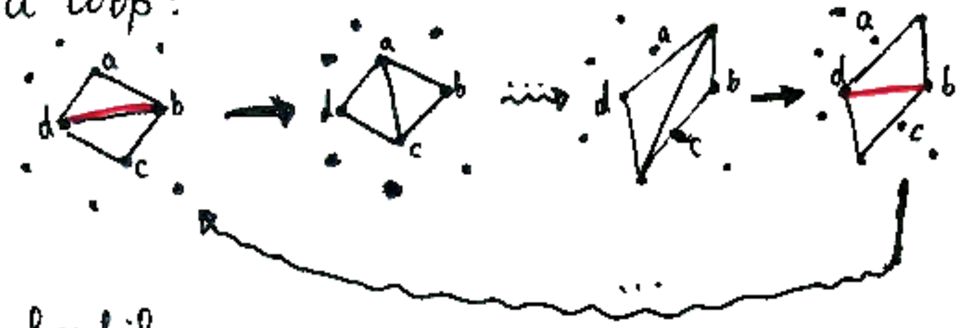
$\mathcal{T}$:= an arbitrary triangulation of S (say the scan triangulation)

While $\exists$ adjacent $T, T' \in \mathcal{T}$ whose vertices are in convex position but C_T contains a vertex of T' do



If the algorithm ever terminates, then it produces indeed a Delaunay triangulation, in view of Corollary 20 and Lemma 21.

Now we move on to prove termination. This is not-so-trivial: ~~because~~ what if the flipped edge reappears later and causes a loop?

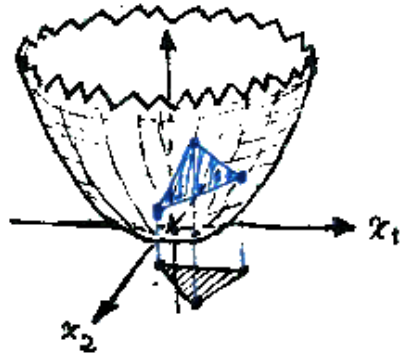


Luckily, the lemma below excludes this possibility.

Lemma 22.
In the Lawson flip algorithm, a flipped edge never appear again.

Corollary 23.
The Lawson flip algorithm terminates after at most $\binom{n}{2} = O(n^2)$ flips. Hence every non-collinear point set S admits a Delaunay triangulation.

Proof of Lemma 22.
Define a "lifting map" $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x) := x_1^2 + x_2^2$, which lifts a point ~~on the plane~~ in S to a point on 3-D paraboloid:



After the lifting, we connect the lifted points by ~~planes~~ triangles corresponding to the plane triangles. So the overall picture would be a continuous 3D surface consisting of piecewise 3D triangles. The projection of these

also, it is bijective from $\mathbb{R}^2$. So given a plane coord., there's only one corresponding surface point.

3D triangles would be exactly the plane triangles. The key observation is:

- Need a flip $\iff$ The corresponding 3D triangles protrude ~~downwards~~ upwards

- Don't need a flip $\iff$ The corresponding 3D triangles protrude downwards

To see this, note that $x \in C_T \iff \varphi(x)$ is simply below the plane defined by the lifted T.

With this observation, the lemma is almost immediate: our blue surface always grows downwards during Lawson flip and remains continuous all the time. So there is no way back. ■

So far, we haven't argued quantitatively why a Delaunay triangulation should look nice.

The next theorem accounts for this:

Theorem 24.
Let $(\alpha_1, \alpha_2, \cdots, \alpha_t) =: \alpha(\mathcal{T})$ be the sequence of interior angles of triangles in $\mathcal{T}$, sorted increasingly. Let $\mathcal{T}$ be a Delaunay triangulation of S (which is assumed in general position, i.e. no cocircular points), and $\mathcal{T}'$ be an arbitrary triangulation of S. Then

$$\alpha(\mathcal{T}) \leq \alpha(\mathcal{T}')$$

in the lexicographic ordering.

Proof. We prove that each flip operation could only decrease $\alpha(\mathcal{T})$ in lexicographic order, from which the theorem follows.

[Flip from | to —]

- Before: $\beta, \theta, \gamma, \varphi, \bar\beta+\bar\theta, \bar\varphi+\bar\gamma$
- After: $\bar\beta, \bar\theta, \bar\gamma, \bar\varphi, \beta+\gamma, \theta+\varphi$

where $\beta<\bar\beta$, $\theta<\bar\theta$, $\gamma<\bar\gamma$, $\varphi<\bar\varphi$.
(Prove it by inscribed angle theorem!)

Note that the smallest angle before the flip among the listed ones is either β, θ, γ or φ. But all angles after the flip are larger than these, so $\alpha(\mathcal{T}_{after}) > \alpha(\mathcal{T}_{before})$. ■

We conclude this section with another nice property of Delaunay triangulations:

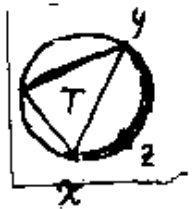
Theorem 25.
Every Delaunay triangulation $\mathcal{T}$ of $S \subseteq \mathbb{R}^2$ contains all Euclidean minimum spanning tree(s) of S.

Note. An Euclidean minimum spanning tree of S is a straightline tree connecting all points in S such that the total length is minimised.

Before proving the result, we derive a handy lemma:

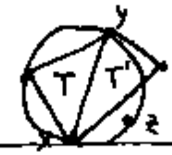
Lemma 26.
Given $S \subseteq \mathbb{R}^2$ and $x, y \in S$. The ~~edge~~ segment $\overline{xy}$ is contained in all Delaunay triangulations of S $\Leftrightarrow$ $\exists$ a circle through x and y (and no other points!) whose interior is empty.

Proof. ($\Rightarrow$)

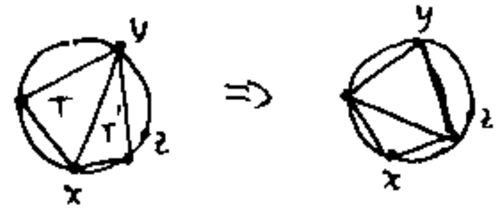


Take an arbitrary Delaunay triangulation $\mathcal{T}$ (which by assumption contains $\overline{xy}$), and single out a triangle $T \in \mathcal{T}$ that has xy as edge. The circumcircle C_T is empty. There can't be a point z lying on the red part of C_T. Suppose to the contrary that there is such z. Then consider the adjacent triangle T' that shares the edge xy:

(1)



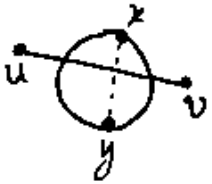
$\Rightarrow z \in C_{T'}$ ↯

(2) 

is also Delaunay but doesn't contain segment $\overline{xy}$. ↯

Hence there can't be such z. But then we are free to "push" C_T to the right while keeping x and y intact. The resulting circle goes through x and y only and has empty interior.

(⇐) We now have at hand a circle C that goes through x and y only and has empty interior. Suppose to the contradition that some Delaunay triangulation $\mathcal{T}$ doesn't contain $\overline{xy}$. Then there must be ~~on~~ a segment $\overline{uv}$ in $\mathcal{T}$ that



crosses $\overline{xy}$ (otherwise, why not adding $\overline{xy}$ back?). Let $\overline{uv}$ be the one that is closest to x. ~~Then~~ And let $T' \in \mathcal{T}$ be the triangle incident to $\overline{uv}$ and lying above.

It should be clear after some thoughts that $\mathcal{T}'$ must be exactly xuv, since putting it elsewhere would lead to a problem. Now observe

$y \in C_{xuv}$, which contradicts with the Delaunay property. ■

Proof of Theorem 25.

Take any edge $\overline{xy}$ from any EMST, and consider a circle C that has diameter $\overline{xy}$:

We claim that C has empty interior and doesn't go through any other points. Once the claim is proved, the theorem follows from Lemma 26.

Now suppose to the contrary that the claim is false:

Assume without loss of generality that z is connected to x via some path that doesn't use $\overline{xy}$. Then we could remove $\overline{xy}$ and add $\overline{yz}$ into the EMST, which leads to a strictly better EMST, a contradiction. ∎