CONVEXITY

Visually, a "round-shape" geometric object is more amenable than a "squiggly" one:

 better than .

Also in the art-gallery setting, a round shape gallery is very easy to guard: we could put a single guard wherever we want, and he sees everything. On the contrary, a squiggly gallery causes much headache.

The concept of convexity captures exactly the "nicer" geometric shapes.

def. Convexity
A set $C \subseteq \mathbb{R}^d$ is called convex if $\forall x, y \in C$, $\overline{xy} \in C$. That is, every two ~~points in C see each other.~~

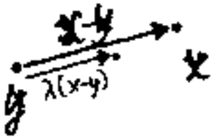
e.g. ,

, $\mathbb{R}^d$, are all convex.

Note that convex sets could be open or unbounded.

Since we will apply algebraic methods extensively, it's good to encode the definition of convexity algebraically:

def.' Convexity (algebraic version)
A set $C \subseteq \mathbb{R}^d$ is called convex if $\forall x, y \in C$, $\forall \lambda \in [0,1]$, we have
$\lambda x + (1-\lambda) y \in C$.



$y + \lambda(x-y)$
$= \lambda x + (1-\lambda) y$.

Usually, it's more convenient to gain extra freedom by considering multiple, instead of 2, points in the definition:

def'' Convexity (algebraic & convenient)

A set $C \subseteq \mathbb{R}^d$ is called convex if any ~~comb~~ "convex combination" of points in C still results in a point in C. That is:
$\forall n \in \mathbb{N}$, $\forall x_1, x_2, \dots, x_n \in C$, we have

$$\sum_{i=1}^{n} \lambda_i x_i \in C$$

whenever $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.

Proposition 5.
Definitions def' and def'' are equivalent.

Proof. ($\Leftarrow$) is trivial since def' is a special case of def''.

($\Rightarrow$) We will later do an algebraic proof, but it is based on a clear geometry intuition: Suppose we want to extend the assertion from 2 points to 3 points, ~~say~~ x_1, x_2, x_3. By def', every y on $\overline{x_1 x_2}$ is also in C. Then

we again apply def' for x_2 and y to conclude that the segment $\overline{x_3 y}$ lies in C. So if we move y along $\overline{x_1 x_2}$, the segment $\overline{x_3 y}$ shall "scan" through the triangle region $x_1 x_2 x_3$, proving that the triangle is contained in C. Similarly, we could lift 3 points to 4 points and so on.

Now we do a pure algebraic proof. By induction on n. The base case $n=2$ is guranteed by def'. For induction step $n \to n+1$:

$$\sum_{i=1}^{n+1} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_{n+1}$$

$$\overset{\text{rescaling}}{=} (1-\lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i + \lambda_{n+1} x_{n+1}$$

Note that $\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} = 1$, so by I.H. we see $\sum_{i=1}^{n} \frac{\lambda_i}{1+\lambda_{n+1}} x_i =: y \in C$. And the equation continues as

$$\dots = (1-\lambda_{n+1}) y + \lambda_{n+1} x_{n+1}$$

$$\in C \quad \text{(by def')}$$

■

While aesthetic value of convexity is impossible to argue about rigorously, there are so many examples showing that "convex = nice" in mathematical contexts. In the following we give several such examples.

Proposition 6
The intersection of arbitrarily many convex sets is still convex. ■

Theorem 7 (Separation theorem)
For any two disjoint compact convex sets $C, D \subseteq \mathbb{R}^d$, there exists a hyperplane separating them, i.e. C lies on one side while D lies on the other.

Proof. Define a distance function $\delta(x,y): C\times D \to \mathbb{R}^+$ by $\delta(x,y) := \|x-y\|^2$. Since $C\cap D = \emptyset$, $\delta(x,y)>0$ everywhere. Note that δ is a continuous function defined on a compact set $C\times D$, hence it attains some minimum value >0 at some $(x^*,y^*)\in C\times D$.

Consider the hyperplane $\mathcal{H}$ that goes through some $z\in \overline{x^*y^*}\setminus\{x^*,y^*\}$ and that is perpendicular to x^*y^*. We claim that $\mathcal{H}$ separates C and D.

Suppose it fails to separate C and D, for the sake of contradiction. W.l.o.g. we may assume $\exists c\in C$ the lies on the other side to x^*.
Then $\overline{x^*c}$ intersects $\mathcal{H}$ at, say, p. By convexity of C, we know $p\in C$ as well.
Now observe the triangle x^*zp. By orthogonality $\angle x^*zp = \pi/2$, so the projection of z onto x^*p lies exactly on $\overline{x^*p}$, which belongs to C by convexity.
Denoting the projection point $\hat{z}$, Clearly $\delta(z,\hat{z}) < \delta(x^*,z)$, hence

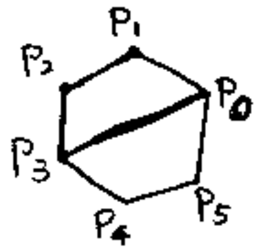
$$\begin{aligned}\delta(y^*,\hat{z}) &\le \delta(y^*,z)+\delta(z,\hat{z})\\ &< \delta(y^*,z)+\delta(x^*,z)\\ &= \delta(x^*,y^*)\end{aligned}$$

a contradiction to minimality. ■

Proposition 8.

Given a convex polygon P with n vertices and a point $x \in \mathbb{R}^2$, one could test in $O(\log n)$ time if $x \in P$.

Proof. This is achieved by a "geometric binary search". Suppose P has vertices $p_0, p_1, \cdots, p_{n-1}$ in ~~circular~~ counterclockwise order. We



cut the polygon into two smaller polygons by an edge $\overline{p_0 p_{\lfloor n/2 \rfloor}}$, and test which side the point x lies. Then we recursively check if x lies in the smaller polygon on that side.

Algorithm InConvexPolygon(x, P)

let $(p_0, \cdots, p_{n-1})$ be the counterclockwise order of vertices of P.

if P is a triangle then

| test directly if $x \in P$ (in constant time)

else

| let P_l be the ~~poly~~ subpolygon on the left to $\overline{p_0 p_{\lfloor n/2 \rfloor}}$

| let P_r be the subpolygon on the right to $\overline{p_0 p_{\lfloor n/2 \rfloor}}$

| if x is on the ~~right~~ left of $\overline{p_0 p_{\lfloor n/2 \rfloor}}$ then

| | InConvexPolygon(x, P_l)

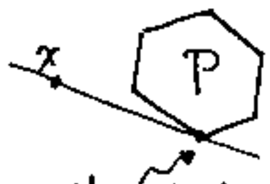
| else

| L InConvexPolygon(x, P_r)

Convexity is crucial here because otherwise we could not guarantee the segment $\overline{p_0 p_{\lfloor n/2 \rfloor}}$ lies in P. ■

Proposition 9

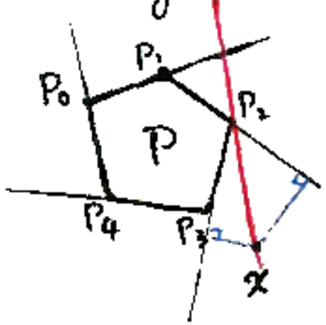
Given a convex polygon P with n vertices and a point $x \in \mathbb{R}^2 \setminus P$, we could find in $O(\log n)$ time the right-tangent point of P with respect to x.



the right-tangent point: all part of P on its left.

Proof. First, a right-tangent point always exists: Theorem 6 ensures that we could separate x and P by a line, so, as we rotate the line gradually we will at some point ~~obtain the~~ encounter a right tangent.

The key observation is the following:



let $(P_0, \dots, P_{n-1})$ be the vertices of P in clockwise order, and consider the rays $P_i \to P_{i+1}$ $(\forall i)$. The rays separate the plane region $\mathbb{R}^2 \setminus P$ into disjoint regions.

P_i is a right-tangent point

$\iff$ x lies inside the region bounded by $P_{i-1} \to P_i$ and $P_i \to P_{i+1}$.

$\iff$ x can be projected onto both $P_{i-1} \to P_i$ and $P_i \to P_{i+1}$, and the sum of projection length is minimised.

With this observation and monotonicity of the property, one could design again a binary search algorithm.

Details are left as an exercise. ■