Duality & Line Arrangement

This section studies some naïve-looking problems related to a finite point set $S \subseteq \mathbb{R}^2$.

- **Collinearity test:** Are there 3 collinear points in S?
- **Minimum area triangle:** From all triangles in $\binom{S}{3}$, find the one with smallest area.

  (Note that the Collinearity test reduces to this problem)
- **Build rotational system:** For every point in S, sort the other points in a clockwise order.

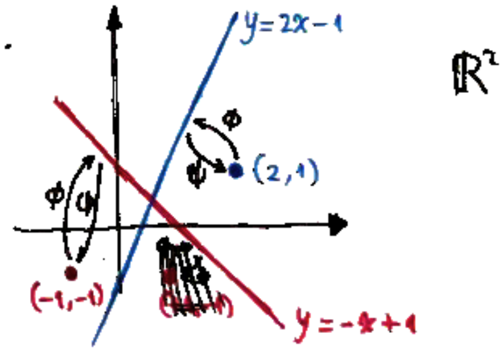
All these problems can be trivially solved in $O(n^3)$ time. But we could do better.

Via a surprisingly simple duality tranform to be explained later, the problems translate to their "dual form" concerning with lines (instead of points) in $\mathbb{R}^2$. It turns out that the dual problems are solvable in $O(n^2)$ using the theory of line arrangements.

def duality maps:

$$\begin{array}{ccc} \text{point} & \xrightarrow{\ \phi\ } & \text{line} \\ (a,b)\in\mathbb{R}^2 & \xleftarrow[\ \psi\ ]{} & \ell: y = ax - b \end{array}$$

e.g.



We remind the readers of a pitfall: A line ℓ could be regarded as a monolithic object or as a set of points, So both $\psi(\ell)$ and $\phi(\ell)$ are well-defined mathematically. However, keep in mind that $\psi(\ell) \neq \phi(\ell)$!

this gives you a point — this gives you a family of lines

So it is important to remember that, when ~~we~~ people say "the dual of ℓ" in the literature, they are referring to $\psi(\ell)$, i.e. treating ℓ as a monolithic body.

You might be confused why we defined duality maps so arbitrarily — there seems to be little geometric correlation between p and $\phi(p)$. But the lemma below actually extracts some rather handy relations between primal and dual.

Lemma 31.

(1) Points p and q have the same x-coordinate
$\iff$ lines $\phi(p)$ and $\phi(q)$ are parallel.

Moreover, when p, q have the same x-coord, $d(p,q) = d(\phi(p), \phi(q))$. ← vertical dist.

(2) p is on ℓ $\iff$ $\psi(\ell)$ is on $\phi(p)$

(3) p is above ℓ $\iff$ $\psi(\ell)$ is above $\phi(p)$

(4) The vertical distance ~~between~~ from p ~~and~~ to ℓ = the vertical distance from $\psi(\ell)$ to $\phi(p)$.

Proof. (1) Exercise.

(2) Assume $p = (a,b)$ and $\ell : y = cx - d$.

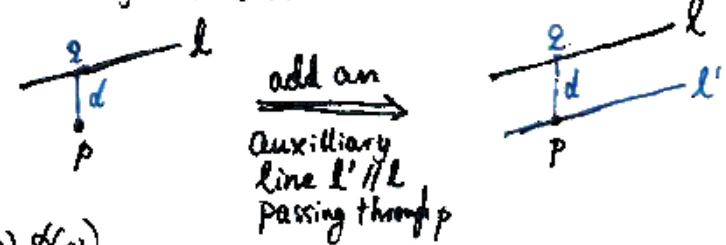
Then $\phi(p) : \quad y = ax - b$

$\psi(\ell) : \quad (c, d)$.

Hence, p is on $\ell \iff b = ac - d$
$\iff d = ac - b$
$\iff \psi(\ell)$ is on $\phi(p)$.

(3) Exercise.

(4) This claim nicely illustrates the use of (1) (2).



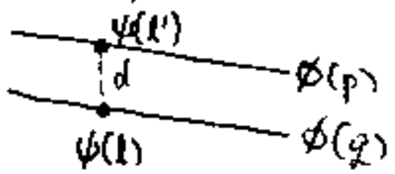
Now look at the dual picture consisting of $\phi(p)$, $\phi(q)$, $\psi(\ell)$ and $\psi(\ell')$. From (1) we have

- $\phi(p) \,/\!/\, \phi(q)$.
- $\psi(\ell)$ and $\psi(\ell')$ have the same x coordinate. Moreover, their distance is exactly d.

From (2) we know

- $\psi(\ell)$ is on $\phi(q)$.
- $\psi(\ell')$ is on $\phi(p)$.

So the picture looks like



■

To test your understanding of duality and the lemma, do the exercise below:

Exercise. Let $I \subseteq \mathbb{R}^2$ be a line segment (regarded as a set of points). Describe the dual $\phi(I)$ in geometry language.

With the guarantee of Lemma 31, it is natural to transform the given point set S to its dual $\phi(S)$, and <u>hope</u> that the geometry properties of interest in S also translate into some corresponding geometry properties in $\phi(S)$. We will illustrate the idea for the problems in our list.

- **Collinearity test.**

  There are 3 collinear points in S

  $\Leftrightarrow$ $\exists$ line ℓ and points $p, q, r \in S$: p, q, r are on line ℓ

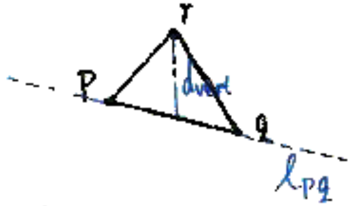
  $\overset{\text{Lemma 31(2)}}{\Longleftrightarrow}$ $\exists$ point t and lines $\ell_1, \ell_2, \ell_3 \in \phi(S)$: t is on ℓ_1, ℓ_2, ℓ_3

  $\Leftrightarrow$ There are 3 lines in $\phi(S)$ that intersect at one point

Therefore, the problem transforms into detecting whether there exist 3 lines in $\phi(S)$ that jointly intersect one point.

- **Minimum area triangle**

For any triangle $pqr \in \binom{S}{3}$ we have



$$\text{area}(pqr) = \frac{|x_p - x_q|}{2} \cdot d_{vert}(r, l_{pq})$$

$$\overset{\text{Lemma } 31(4)}{=} \frac{|x_p - x_q|}{2} \cdot d_{vert}(\psi(l_{pq}), \phi(r))$$

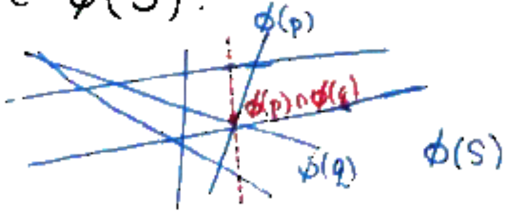
$$= \frac{|x_p - x_q|}{2} \cdot d_{vert}(\phi(p) \cap \phi(q), \phi(r))$$

where $\psi(l_{pq}) = \phi(p) \cap \phi(q)$ because both p and q are on l_{pq} and thus by Lemma 31(2), $\psi(l_{pq})$ must be on both $\phi(p)$ and $\phi(q)$.

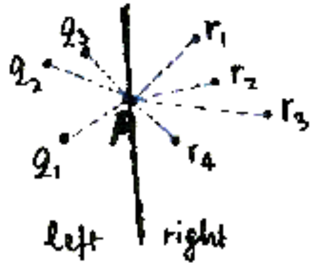
So the problem reduces to computing

$$\min_{r \in S} d_{vert}(\phi(p) \cap \phi(q), \phi(r))$$

for every pair $p, q \in S$. This is a purely geometry problem in the dual picture $\phi(S)$.



- **Build rotational system**

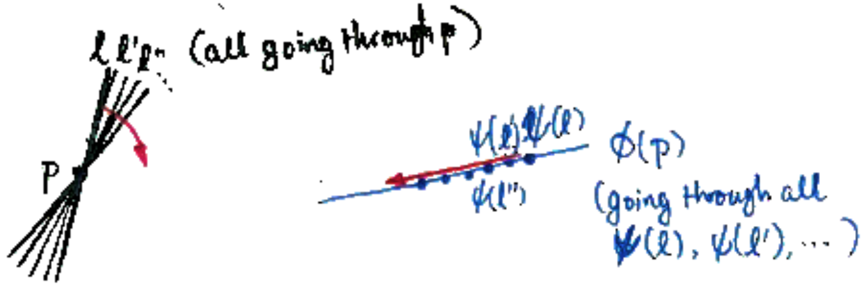


Split $S \setminus \{p\}$ into left part $\{q_1, \dots, q_k\}$ and right part $\{r_1, \dots, r_t\}$. Imagine rotating the vertical line clockwise around p, until it is vertical again. During the rotation it will sequentially hit $(p_1, p_2, \dots, p_{n-1})$. Although the q's and r's might be interlacing in the sequence, the relative order inside the q's (resp. the r's) are preserved. e.g.

$$(r_1, q_1, r_2, r_3, q_2, r_4, q_3)$$

Hence, once we get the sequence $(p_1, \dots, p_{n-1})$, it's straightforward to recover the rotational system clockwise: $(r_1, r_2, \dots, r_t, q_1, q_2, \dots, q_k)$.

But the "rotation & hit" procedure has a direct correspondence in the dual picture $\phi(S)$. Rotating a line around p corresponds to moving a point along $\phi(p)$. Since we start with



slope $+\infty$ and ends at slope $-\infty$, the point movement is from the right to the left in the dual.

With this observation, computing $(p_1, \cdots, p_{n-1})$ amounts to ~~finding~~ reading the intersections of $\phi(p)$ with other dual lines from right to left.

However, the above discussions do not yield directly $O(n^2)$ algorithms for our purposes. To this end, we need to inspect in more detail the structure of the dual picture, or more generally, the arrangement of n lines in $\mathbb{R}^2$.

Given a set $\mathcal{L}$ of n lines in $\mathbb{R}^2$, we may naturally invent the notions of "vertex", "edge" and "face" just as we did for Voronoi diagrams. The entire incidence structure is called "line arrangement", denoted $\mathcal{A}(\mathcal{L})$.

whole structure $= \mathcal{A}(\mathcal{L})$

Lemma 32.
Any line arrangement $\mathcal{A}(\mathcal{L})$ contains at most $\binom{n}{2}$ vertices, n^2 edges and $\binom{n+1}{2}+1$ faces.

Proof. Without loss of generality we assume that no two lines in $\mathcal{L}$ are parallel, and no three ~~points~~ lines in $\mathcal{L}$ intersect at one point. (We could alway perturb the lines a bit to fulfil the requirements and the counts could only increase)

Every two lines intersect exactly once, hence #vertices $= \binom{n}{2}$. Every line intersect $(n-1)$ others, hence is divided into n edges. So #edges $= n \cdot n = n^2$. Finally, we build a planar graph on top of $\mathcal{A}(\mathcal{L})$ by introducing an "infinite vertex" which connects to all infinite edges. By Euler's formula,

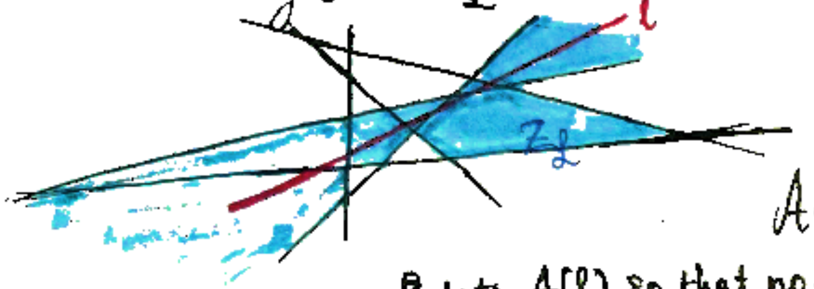
$$(\#\text{vertices}+1) - (\#\text{edges}) + (\#\text{faces}) = 2$$

$$\Rightarrow \#\text{faces} = 2 - \left(\binom{n}{2}+1 \right) + n^2$$
$$= n^2 - \binom{n}{2} + 1$$
$$= \binom{n+1}{2} + 1. \qquad \blacksquare$$

So ~~the~~ line arrangement is not a super simple object to deal with. But still we hope for the best: Can we construct it in (optimal) $O(n^2)$ time? The answer is yes, and the whole magic hides in the ~~fact that lines are straight~~ theorem below. From a high level, lines are straight and they display quite "linear" behaviour when it comes to incidence structure.

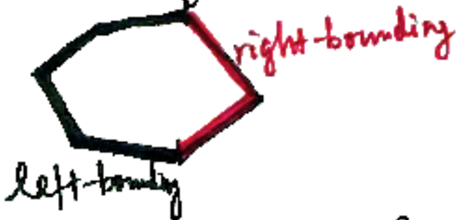
~~Theorem 33 (Zone theorem) For any line arrangement $\mathcal{A}(\mathcal{L})$ and line ℓ, ℓ could be incident to at most~~

Theorem 33. (Zone theorem)
Let $A(\mathcal{L})$ be a line arrangement and ℓ be a line. Denote by $Z_{\mathcal{L}}$ the collection of faces in $A(\mathcal{L})$ that ℓ intersects. ("Z" for "zone") Then the total number of edges in $Z_{\mathcal{L}} \leq 10n$.



Rotate $A(\mathcal{L})$ so that no line is horizontal.

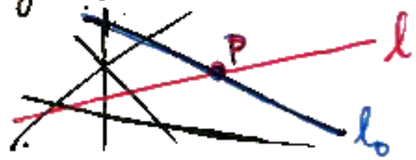
Proof. For a face f and an edge $e \in \partial f$, we call e left-bounding (to f) if f is completely at its right. Similarly define the notion of right-bounding.



Note that any $e \in \partial f$ is either left- or right-bounding because f is a convex polygon in our context.

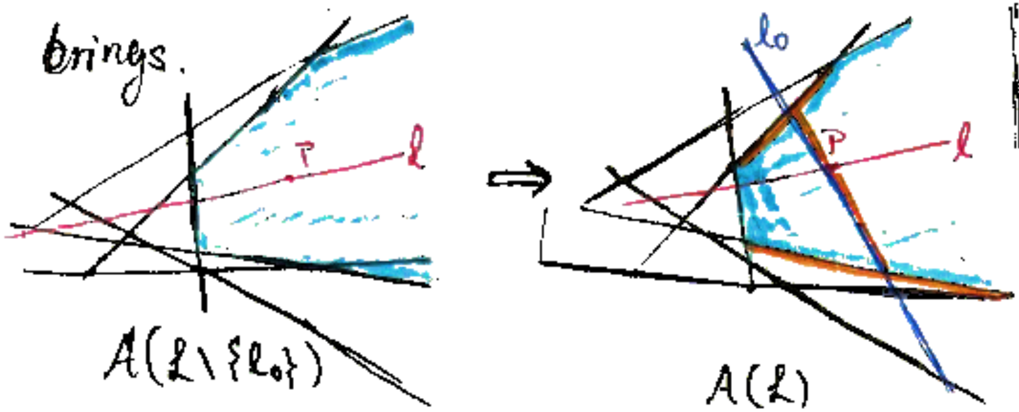
We shall prove by induction that the number of left-bounding edges of faces in $Z_{\mathcal{L}}$ is at most $5n$. A symmetric argument applies to right-bounding edges, so altogether we get the claimed $10n$ upper bound.

When $n=1$ ~~then~~ the claim is trivial. Now we proceed from $n-1$ to n. Let p be the rightmost intersection of ℓ and $\mathcal{L}$. Assume for the moment that there is only one line, say ℓ_0, in $\mathcal{L}$ that goes through p.



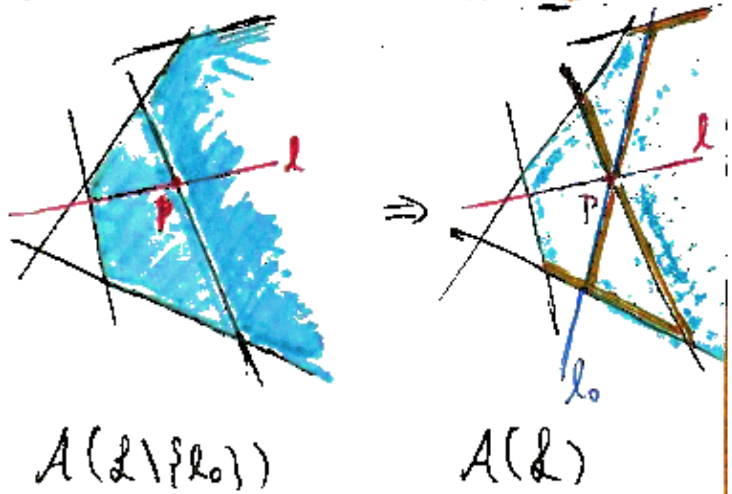
By induction hypothesis, the total number of leftbounding edges of $Z_{\mathcal{L} \setminus \{\ell_0\}}$ is at most $5(n-1)$.

Now we add ℓ_0 back and discuss how many new left-bounding edges it brings.



Observe that l_0 could only go through the rightmost face in $Z_{L\setminus\{l_0\}}$ (shaded blue in the illustration). Since the face is convex, l_0 intersects it at most twice. Each intersection could split a left-bounding edge into two. Plus that l_0 itself could be left-bounding, at most 3 new left-bounding edges are added, and then $\# \le 5(n-1)+3 \le 5n$.

With this experience, we could now remove our artificial assumption that l_0 is the only line going through p.



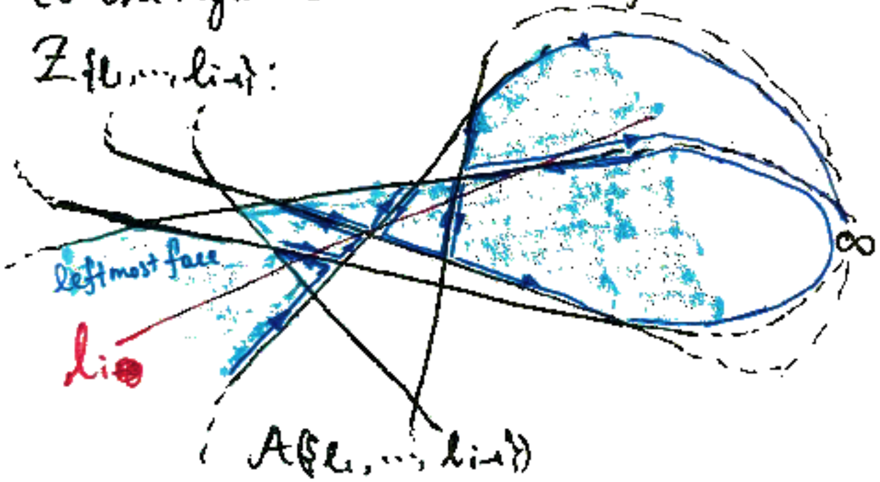
This time, the potential influence spreads over two faces in Z. Still, we have two intersections with one on each side. Plus an additional split at point p and two more edges induced by l_0 itself. So the total number of addition is 5, then $\# \le 5(n-1)+5 \le 5n$. ■

Corollary 34.

One may build $A(L)$ in $O(n^2)$ time.

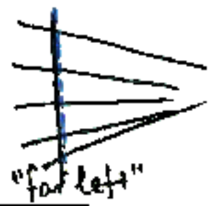
Proof. Assume $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$. We incrementally build $A(\{\ell_1\}), A(\{\ell_1, \ell_2\}), \dots, A(\mathcal{L})$ by inserting one line at a time. At step i we insert ℓ_i into $A(\{\ell_1, \dots, \ell_{i-1}\})$.

Suppose for now that we could find the leftmost face in $Z(\{\ell_1, \dots, \ell_{i-1}\})$ that ℓ_i intersects. Then we could walk systematically to the right to traverse all faces in $Z\{\ell_1, \dots, \ell_{i-1}\}$:



To be specific, we walk along the boundary of the current face in counterclockwise direction, until we detect an intersection with ℓ_i. Then we create a new vertex at the intersection and split the edge & face accordingly. Next we "jump" to the other side of the edge to enter a new face. Repeat until no more intersection could be detected.

By zone theorem, our traversal takes time $5i = O(n)$. So the only missing piece is to show that we could indeed locate the leftmost face in $Z\{\ell_1, \dots, \ell_{i-1}\}$. But this is easy due to the following observation: In the "far left" the lines are sorted by their slope; the ~~higher~~ larger slope it has, the lower the line lives. So the location of ℓ_i could be pinned down in even $O(\log n)$ time. ■

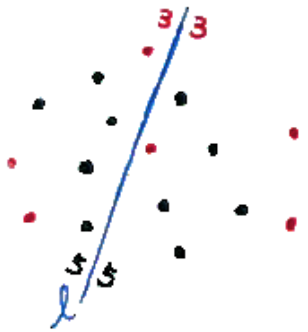


Therefore, all three problems in our list can be effectively solved in $O(n^2)$ time by working in the dual picture!

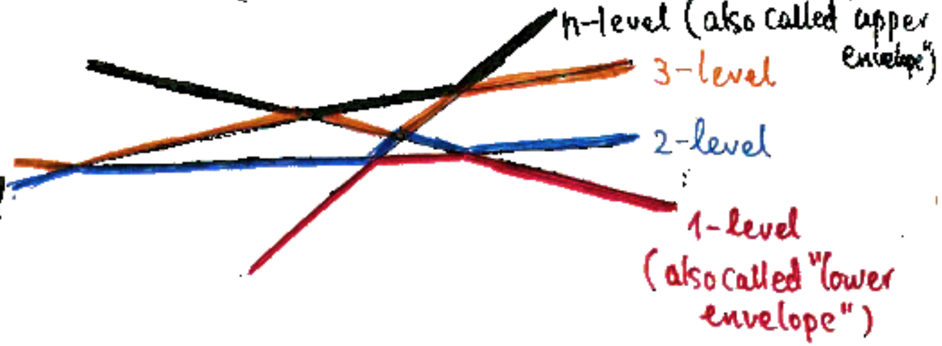
(Details are left as an exercise.)

We conclude the Section by a beautiful theorem which displays, once again, the power of duality.

Theorem 35. (Discrete Ham Sandwich Thm)
For any two finite points sets $S, T \subseteq \mathbb{R}^2$, there exists a line ℓ that bisects both S and T. That is $|S \cap \ell^-|, |S \cap \ell^+| \leq |S|/2$ and $|T \cap \ell^-|, |T \cap \ell^+| \leq |T|/2$.



def. k-level.
The k-level of $A(\mathcal{L})$ is the collection of edges that have $(k-1)$ lines below and $(n-k)$ lines above.



Proof. Note that it suffices to prove the theorem for odd $|S|, |T|$ (when they are even we may remove an arbitrary point and apply the result for odd sizes)

Also, assume without loss of generality that no points in $S \cup T$ have the same x-coordinate. (Otherwise, rotate the system infinitesimally.)

Then we may translate the condition to dual as follows:

$$\exists \ell : \begin{cases} \leq \frac{|S|-1}{2} \text{ points in } S \text{ below/above } \ell \\ \leq \frac{|T|-1}{2} \text{ points in } T \text{ below/above } \ell \end{cases}$$

Lemma 31(3)

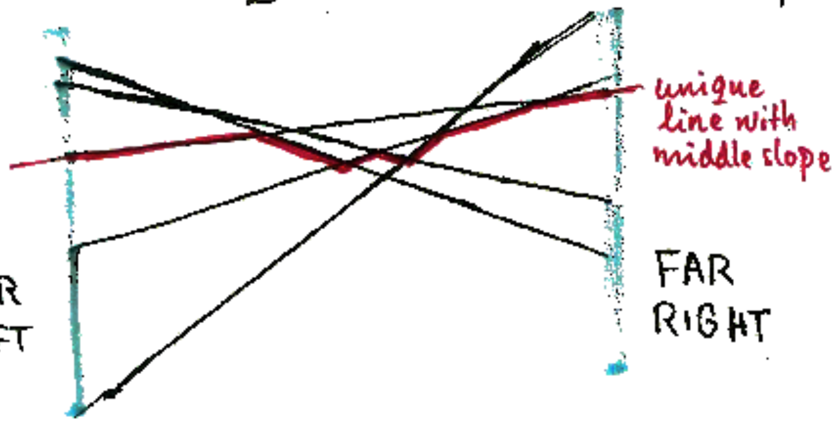
$\Longleftrightarrow \exists p: \begin{cases} \le \frac{|S|-1}{2} \text{ lines in } \phi(S) \text{ below/above } p \\ \le \frac{|T|-1}{2} \text{ lines in } \phi(S) \text{ below/above } p \end{cases}$

$\Longleftrightarrow \exists p:$ p is on the mid-level of $\phi(S)$ and also the mid-level of $\phi(T)$

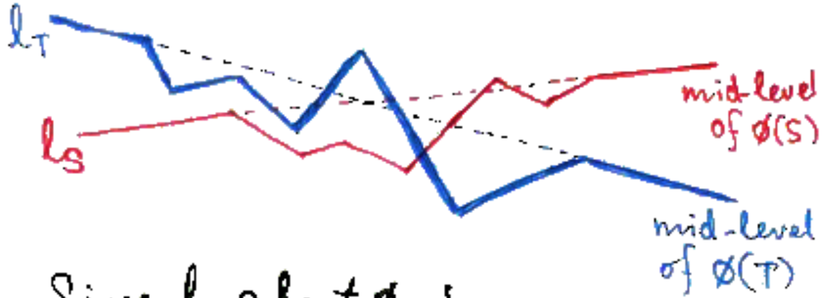
$\Longleftrightarrow$ (mid-level of $\phi(S)$) $\cap$ (mid-level of $\phi(T)$) $\neq \emptyset$

Recall the in the "far left" and "far right", the lines are sorted according to slope.

Therefore, the left ~~extreme~~ and right ends of the mid-level of $\phi(S)$ (resp. $\phi(T)$) are both the unique line with middle slope.



So the middle levels of $\phi(S)$ and $\phi(T)$ have the shape



Since $l_T \cap l_S \neq \emptyset$, by continuity we know (mid-level of $\phi(S)$) $\cap$ (mid-level of $\phi(T)$) $\neq \emptyset$, proving the theorem. ∎

Remark. The theorem is a discrete specialisation of a much stronger Ham Sandwich theorem, stating that there exists a hyperplane that bisects both $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^n$. Here "bisection" could be taken with respect to any continuous measure.