POLYGONS

Starting from this section we will see a major theme change. What we have discussed up to now are mostly graph-theoretical results, meaning that we concerned about the geometric properties of some <u>abtract graph structure</u>.

However, in the following we shall move to <u>pure geometry objects</u> without underlying graph concepts. Accompanied with this, we will apply more algebraic tools since they are indispensible when formalising geometry (especially in high dimensional space). But don't worry: the current section is just a mild transition that familiarise you with our new theme.

Triangles are perhaps the most basic geometric shape in $\mathbb{R}^2$, and they naturally generalise to a more complex object called polygon. As we already know in elementary school (as well as by intuition), polygons are quite expressive to model daily shapes, hence worth a close study. Our main point in this section is: Polygons are also easy to handle, in the sense that they decompose into triangles.

Let's start with a formal definition of polygons:

def. polygon.
A polygon P is a compact region whose boundary ∂P is a Jordan curve consisting of finitely many consecutive line segments.

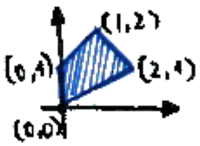
e.g.  are polygons;

are not.

People usually call the boundary segments "edges", and their endpoints "vertices". But it is worth remembering that ~~these~~ a polygon is by definition a <u>region</u>, which is <u>purely geometric</u> and has nothing to do with a graph.

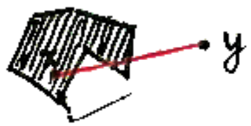
By Jordan curve theorem, a polygon can be recovered if we are given only its boundary segments. Hence in practice people just <u>represent</u> a polygon concisely by listing all the vertices in order.
i.e ~~the~~ coordinates

e.g. $(0,0)$, $(0,1)$, $(1,2)$, $(2,1)$
represents



Question. Given a polygon P (in the representation ~~[illegible]~~ discussed above) and a point $x \in \mathbb{R}^2$, how do we test if $x \in P$?

Solution.
Idea: Find an arbitrary point $y \notin P$ (say a point lying to the right of the rightmost vertex in P). Count the number of intersections of segment $\overline{xy}$ with ∂P. Odd $\Rightarrow x \in P$; even $\Rightarrow x \notin P$.



Be careful: what if $\overline{xy}$ overlaps with some boundary segment? what if $\overline{xy}$ goes through a vertex? Then the test doesn't work. So we should choose y carefully to avoid these degeneracies.

Exercise. Explicitly state the algorithm, prove its correctness, and analyse the run-time.

Now we enter the real part of the story: polygon triangulation.

def triangulation.
A polygon triangulation is a decomposition

of a given polygon into "disjoint" union of triangles, such that the vertices of triangles are exactly vertices of the polygon. Here "disjoint" means every two triangles don't intersect or only intersect at a vertex/ an edge.

More formally, let P be a polygon. A triangulation of P is a finite collection $\mathcal{T}$ of triangles s.t.

(1) $\bigcup_{T\in\mathcal{T}} T = P$

(2) $\forall T \neq T' \in \mathcal{T}$, $T \cap T' = \emptyset$, a vertex, or an edge shared by T & T'.

(3) $\cancel{\forall}\bigcup_{T\in\mathcal{T}} V(T) = V(P)$

vertex set of T

vertex set of P

Remark. It might be tempting to think of other plausible ways to formalise the same concept. Unfortunately these are very likely wrong. For example, if we change (2) into

(2)$^?$ $\forall T \neq T' \in \mathcal{T}$, $|T \cap T'|$ has Lebesgue measure 0

then we will encounter the problem of "sharing half edge":

This gives us a weaker definition of triangulation, which is okay for some applications, but not very convenient in general.

[redacted] Similarly, if we change (3) into

(3)$^?$ $\forall T \in \mathcal{T}$, $V(T) \subseteq V(P)$

then things become awry when we have collinear vertices:

Theoretically, one could simply prohibit collinear vertices in polygons. But in practice, it would be much more convenient to allow them.

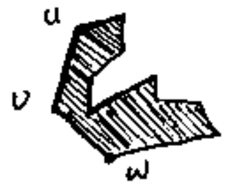
Theorem 1.

Every polygon admits a triangulation.

Proof. By induction on the number of vertices, n. When $n=3$, the polygon itself is a triangle and of course admits a triangulation ($\mathcal{T} := \{\text{itself}\}$). Next we do the induction step $n \to (n+1)$.

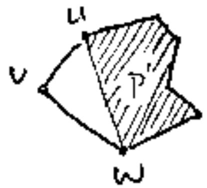
Denote P the polygon we are considering. Let v be the leftmost vertex of P; if there are multiple leftmost vertices, choose the bottom one. Assume the neighbouring vertices of v are u and w.

Note that by our choice of v, uvw can't be collinear.



We distinguish two cases.

(a) If $\overline{uw} \cap \partial P = \{u, w\}$ (i.e. $\overline{uw}$ lies inside P), then we consider the polygon formed P' by the "right-hand-side" of $\overline{uw}$.



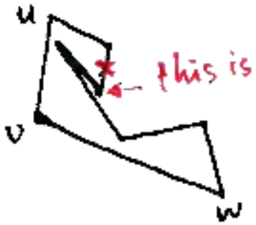
Formally, $V(P') := V(P) \setminus \{v\}$. By induction hypothesis, P' admits a triangulation, say $\mathcal{T}'$.

It's not hard to check that $\mathcal{T} := \mathcal{T}' \cup \{uvw\}$ is a legal triangulation of P.

(b) If $\overline{uw} \cap \partial P \neq \{u, w\}$, then there must be a vertex lying inside the triangle uvw. Let x be the one that is farthest away from line uw among these vertices.

It should be clear that $\overline{vx} \cap \partial P = \{v, x\}$. So, following essentially the same spirit of (a), we consider the two polygons separated by $\overline{vx}$ and get respective triangulations $\mathcal{T}'$ and $\mathcal{T}''$. $\mathcal{T} := \mathcal{T}' \cup \mathcal{T}''$ is then a triangulation for P. ∎

Remark. In case (b), we could not select x as the closest vertex to v; for counter example:



this is closest to v, but clearly $\overline{xv} \cap \partial P \neq \{x, v\}$.

However, we indeed could choose x so as to maximise the distance to some ℓ other than uw. That is, the statement "farthest away from line uw" could be replaced by "farthest away from line ℓ", for any fixed ℓ.

The proof of Theorem 1 directly translates into an $O(n^2)$ algorithm of triangulating a given polygon. There exist linear time algorithms, though they are quite involved.

Theorem 1 convinces ourselves that polygons are indeed simple to cope with by computers. For instance, we could compute the area of a polygon by summing up the areas of triangles in its triangulation.

Proposition 2

Every triangulation of polygon P with n vertices contains exactly $(n-2)$ triangles and $(2n-3)$ edges.

Proof. Easy induction. ■

Corollary 3

The interior angles of a polygon with n vertices sum up to $(n-2)\pi$.

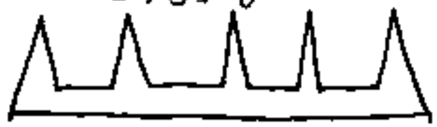
We close this section by a very elegant application of triangulation on the so-called "art gallery problem".

Question: We model the floor plan of an art gallery by a polygon P. We want to place one or more "guards" in P so that they could see every corner of the gallery.

Formally, ~~we say~~ for $x, y \in P$ we say "x and y see each other" if $\overline{xy} \in P$. We are searching for a point set $X \subseteq P$ s.t. $\forall y \in P, \exists x \in X$: x and y see each other.

~~What is the~~ How many guards are necessary to guard a gallery with n vertices? How many are sufficient?

It's not hard to come up with an example that $\lfloor n/3 \rfloor$ guards are necessary in general:

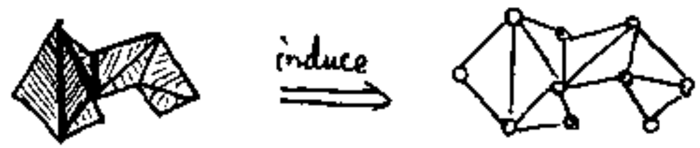


each "tooth" would require a separate guard.

The amazing result we are going to introduce asserts that $\lfloor n/3 \rfloor$ guards always suffice. It was first proved by Chvátal, but Fisk later gave a simple and beautiful proof using triangulation.

Theorem 4.
For ~~all~~ any polygon P with n vertices, there exists a guard set $X \subseteq P$ with $|X| \leq \lfloor n/3 \rfloor$.

Proof. Take an arbitrary triangulation $\mathcal{T}$ of polygon P. Consider a graph G induced by the edges in $\mathcal{T}$.



P and $\mathcal{T}$ $\qquad$ G.

One could prove easily by induction that G is 3-colourable. ~~By using~~ Or, if you don't like induction, simply observe that the dual graph of G must be a tree, hence we could 3-colour G greedily.

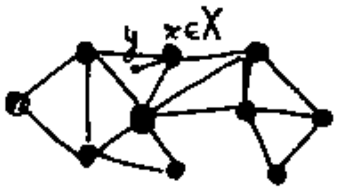
Now fix a 3-colouring of G. It

partitions the vertices into three disjoint sets (each set has the same colour), and at least one of them has size $\lfloor n/3 \rfloor$. Let's call it X.

- We claim that X ~~is exactly the~~ is actually a guard set, because:

  $\forall y \in P,\ \exists T \in \mathcal{T} : y \in T$.

  ~~Note~~ But the 3 vertices of T has distinct colours, hence $T \cap X \neq \emptyset$. Hence $\exists x \in X$ s.t. ~~x and y see~~ $\overline{xy} \in T \subseteq P$; that is, x and y see each other. ■



$X = \{\text{red vertices}\}$

- As a final remark, although $\lfloor n/3 \rfloor$ guards are needed in general, in many cases we could do much better, for instance a convex polygon (which will be defined rigorously in the next section) needs only one guard. In fact, deciding the following problem is NP-hard and $\exists\mathbb{R}$-complete:

  "Given a polygon P and a number $k \in \mathbb{N}$, Can P be guarded by $\leq k$ guards?"

  It is not even known if the problem is in NP — since Abrahamsen, Adamaszek & Miltzow constructed an example where all vertices have integer coordinates yet the optimal solution will put a guard at irrational coordinate; and irrational numbers cannot be represented succinctly in a straightforward way.