Crossing Number

Planarity is an important topic but not the entire story. There are many interesting graphs that are non-planar, yet we want to draw them neatly on the plane $\mathbb{R}^2$. So it's time to extend our notion of plane graph (i.e. embedding) to a more general term:

def drawing.
A drawing of an abstract graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is a tuple (V, E), where ~~$V \subseteq \mathbb{R}^2$ is the point set and $E \subseteq \mathbb{R}^2$ is the edge set~~

(1) $V \subseteq \mathbb{R}^2$ is a finite set of points that corresponds to $\tilde{V}$.

(2) $E \subseteq \mathbb{R}^2$ is a finite set of arcs whose endpoints are in V, and v_1, v_2 are connected by an arc iff $\tilde{v}_1, \tilde{v}_2$ are adjacent in $\tilde{E}$.

(3) $\forall$ distinct $e, e', e'' \in E$, $\mathring{e} \cap \mathring{e}' \cap \mathring{e}'' = \emptyset$. That is, no three edges cross at the same point.

It should be clear that every graph G admits a drawing.

Now we could ask: what is the best possible way to draw a graph? Well, naturally, "best" here means "as few crossings as possible", because fewer crossing leads to lower visual complexity. This question motivates the definitions below.

def optimal drawing.
An optimal drawing of G is a drawing that minises the number of crossings.

Note that an optimal drawing always exists, since it's pretty trivial to come up with a drawing with <u>finitely</u> many crossings, and thus the "minise" makes sense.

def crossing number $Cr(G)$:
\# crossings in any optimal drawing of G.

For planar graphs G, clearly $Cr(G) = 0$. And the converse is also true obviously.

eg. Crossing number of K_5.

Since K_5 is non-planar, we have $Cr(K_5)>0$. On the other hand, we could draw K_5 as

, so $cr(K_5)\le 1$, and thus $cr(K_5)=1$.

As an appetizer, we provide a simple lower bound for $Cr(G)$.

Lemma 24

$Cr(G) \ge |E| - (3|V|-6)$.

proof. We start from an optimal drawing of G and remove crossings from it. Whenever there remains a crossing, we pick any edge that causes the crossing and remove it. Finally we arrive at a planar graph $G'=(V,E')$, on which Euler's Formula holds.

So $\begin{cases} |E'| = 3|V|-6 \ ; \\ |E|-|E'| \le cr(G) \end{cases}$ and the lemma follows. ■

In every round we remove an edge and at least one crossing is gone, so we have at most $Cr(G)$ many rounds.

The lemma is loose when the graph is somewhat dense: Consider a graph with $O(|V|^{3/2})$ edges, say. Intuitively, each edge will cross many others, so in the first few rounds, a lot of crossings are removed in each round but we only counted one in the proof. Then the inequality $|E|-|E'| \le cr(G)$ would be very loose.

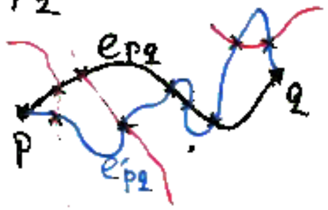
Towards an improved lower bound, we first take a closer look at the property of an optimal drawing.

Proposition 25

In any optimal drawing, no two edges shall share more than one point.

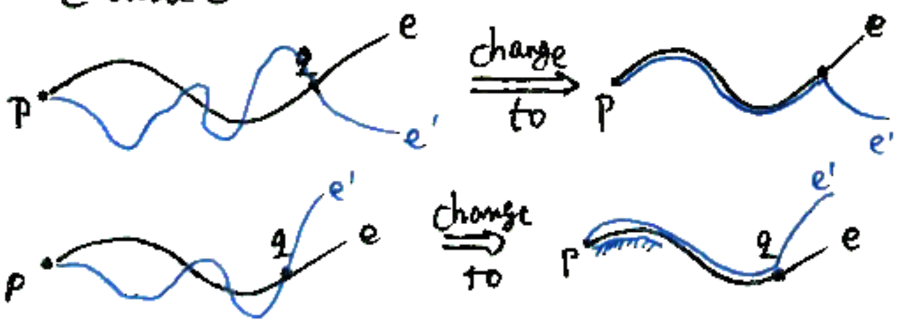
Proof. Suppose to the contradiction that e and e' share two distinct points p and q. Let e_{pq} (resp. e'_{pq})

be the part of e (resp. e') that lies between p and q. Of course, e_{pq} may cross e'_{pq} or some other edges, but it doesn't matter. Assume without loss of generality that e_{pq} has <u>no more crossings than</u> e'_{pq} does.



e_{pq} has 7 crossings
e'_{pq} has 9 crossings

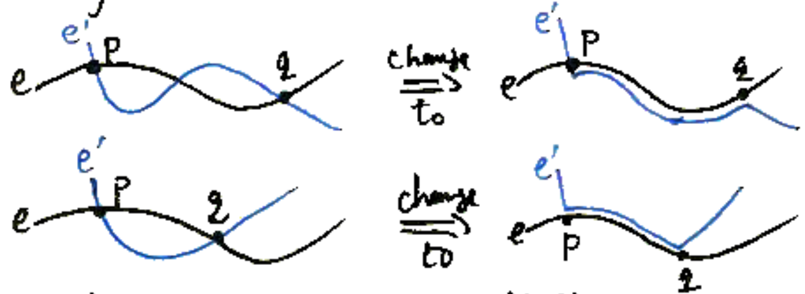
(a) If p is the common endpoint of e and e':



So now e'_{pq} has the same number of crossings as e_{pq} (which is assumed more economic). But we saved a crossing at q. So this contradicts with optimality.

(b) If q is the common endpoint of e and e', the argument is similar.

(c) If p and q are interior points of e and e':



which leads to contradiction, too. ■

With this insight, the following result has an astonishingly simple and elegant proof.

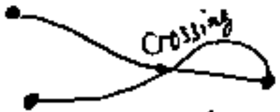
Lemma 26 (Crossing Lemma)
$cr(G) \ge \frac{|E|^3}{64|V|^2}$ if $|E| \ge 4|V|$.

Proof. Consider the random experiment where we choose a random subset $V' \subseteq V$; each vertex is put into V'

with probability p independently.
This ~~generates~~ induces a random subgraph
$G[V'] =: G' =: (V', E')$.

Now it's easy to compute $\mathbb{E}(|V'|) = p \cdot |V|$, and $\mathbb{E}(|E'|) = p^2 |E|$. As for crossing number, we note that each crossing involves exactly 4 vertices:

and the weird configuration



is excluded by ~~Lemma~~ Proposition 25. Therefore

$$\mathbb{E}(cr(G')) = p^4 \cdot cr(G).$$

But we also know that $cr(G') \geq |E'| - (3|V'| - 6)$, hence by taking expectation we obtain

$$p^4 \cdot cr(G) \geq p^2 |E| - (3p \cdot |V| - 6)$$

or simply

$$cr(G) > \frac{|E|}{p^2} - \frac{3|V|}{p^3}.$$

Taking $p := \frac{4|V|}{|E|} \leq 1$ gives the result. ■

Remark. ~~Note that~~ The key intuition behind the proof is that: the random graph G' is sparse because p^2 drops faster than p. So Lemma 24 performs reasonably well on G'. Then we pump this back to the original graph G, circumventing the weakness of Lemma 24.

Up to a constant factor, the Crossing Lemma is tight: Pach and Toth constructed a graph family s.t. $cr(G) \leq \frac{16}{27\pi^2} \cdot \frac{|E|^3}{|V|^2}$.
(Note that it doesn't mean a general upper bound for all graphs!)

To put an end to this section as well as the entire topic on planarity, we expose a nice application ~~that has~~ ~~no less~~ of crossing lemma.

def. incidence

For a finite point set $P \subseteq \mathbb{R}^2$ and a finite line set $\mathcal{L} \subseteq \mathbb{R}^2$, the incidence of system $(P, \mathcal{L})$ ~~denoted inc(P,L)~~ is defined as

$$\mathrm{inc}(P, \mathcal{L}) := \sum_{p \in P} (\# l \in \mathcal{L} : p \in l)$$

$$= \sum_{l \in \mathcal{L}} (\# p \in P : p \in l).$$

e.g.

$\mathrm{inc} = 8$.

Theorem (Szemerédi-Trotter)

For any finite point set P and line set $\mathcal{L}$ in $\mathbb{R}^2$ with $n := |P|$, $m := |\mathcal{L}|$, we have

$$\mathrm{inc}(P, \mathcal{L}) \leq 2^{5/3} \cdot n^{2/3} m^{2/3} + 4n + m.$$

Proof. Define graph $G := (P, E)$, where $pq \in E$ iff $\exists l \in \mathcal{L} : p \in l, q \in l$ and they are consecutive on l.

For example:

Note that $|E| = \sum_{l \in \mathcal{L}} [(\# p \in P : p \in l) - 1]$

$$= \mathrm{inc}(P, \mathcal{L}) - m.$$

So it suffices to upper bound $|E|$. We apply Crossing Lemma to obtain $|E| \leq \sqrt[3]{64 n^2 \cdot \mathrm{cr}(G)}$. But if we look at the specific drawing of G induced by the point-line arrangement, we see that each individual $l \in \mathcal{L}$ could incur at most $(m-1)$ crossings, hence $\mathrm{cr}(G) \leq \frac{m(m-1)}{2}$.

Putting things together we yield

$$\operatorname{inc}(P, \mathcal{L}) \le 2^{5/2} (nm)^{3/2} + m .$$

But be alert that Crossing lemma has a condition which we didn't actually check. When the condition doesn't hold, i.e. $|E| < 4n$, we still have

$$\operatorname{inc}(P, \mathcal{L}) = |E| + m < 4n + m ;$$

that's why there's a "$4n$" in the Theorem. $\blacksquare$