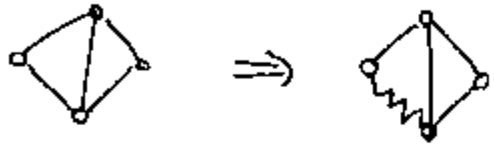
Unique Embedding

Given a planar graph, there are infinitely many ways to embed it in $\mathbb{R}^2$: we could always introduce some "pertubations" to an embedding and modify it to a new one.



But clearly not all these embeddings are "distinct" in some sense. For the two embeddings drawn above, one would argue that they are "essentially the same", or "equivalent", "isomorphic", etc. The thing is, we didn't set a common ground for our understanding of "same", "equivalent", ..., and we will do it next.

~~def. We say two plane graphs H and H'~~
~~are [illegible] com~~

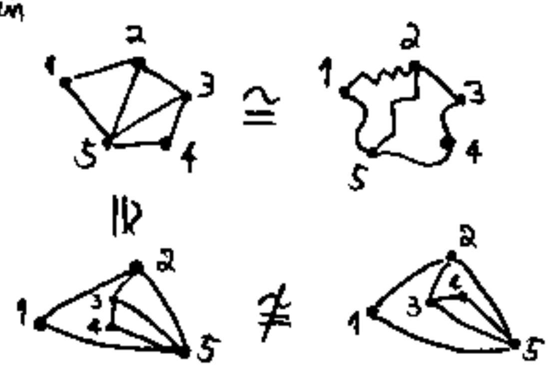
def. plane graph isomorphism.

We say two <u>plane</u> graphs H and H'

are isomorphic, denoted $\mathcal{H} \cong \mathcal{H}'$, if we could find a bijection $\sigma: F(H)\to F(H')$ s.t. $\forall f\in F(H)$, ∂f and $\partial\sigma(f)$ are bounded by exactly the same set of edges.

The bijection σ is thus called the isomorphism between $\mathcal{H}$ and $\mathcal{H}'$.

e.g.



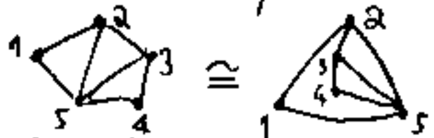
As an easy exercise, please show that isomorphism preserves all incidence relations. More precisely:

Exercise. Assume σ is an isomorphism between $\mathcal{H}$ and $\mathcal{H}'$. Show that

(1) The underlying abstract graph of $\mathcal{H}$ and $\mathcal{H}'$ are the same.

(2) $\forall$ edge e and face $f\in F(H)$, e is on ∂f $\Longleftrightarrow$ e is on $\partial\sigma(f)$.

You might think the definition was too weak because



but they look quite different! Well, it depends on how strict you perceive the difference. If you project them stereographically as in Theorem 10, then probably you will notice that they are essentially the same. Roughly, our definition allows the "flipping inside-out" trick when we massage the embeddings.

There are, of course, stronger notions of isomorphisms. But we shall not pursue them in the notes.

The main theorem of this section states that any 3-connected planar graph admits a unique embedding up to isomorphism. That is, no matter how we draw the graph on $\mathbb{R}^2$ without crossing, the boundary edges of each face is simply fixed.

We need a techical lemma.

Lemma 19.
For a 3-connected planar graph G and a cycle C in it,
C bounds a face in <u>every</u> embedding of G
$\Longleftrightarrow$ C is chordless and $G \setminus C$ is connected.

Proof. The "$\Leftarrow$" part is trivial: In every embedding of G, either all of $G \setminus C$ lies in the interior of C or all of $G \setminus C$ lies in the exterior of C (otherwise $G \setminus C$ would be disconnected), so C bounds a face.

Now we prove "$\Rightarrow$" by contradiction. Assume C bounds a face in every embedding, but (a) C has a chord; or (b) $G \setminus C$ is separated.

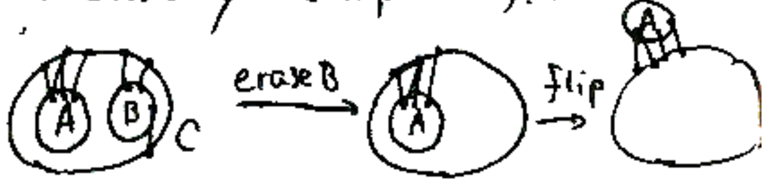
(a)

Without loss of generality assume C bounds the outer face. Then the chord is in the interior of C. But then we could simply drag the chord to the exterior and produce an embedding

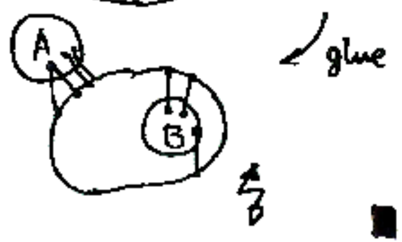
where C doesn't bound the outer face. But it can't bound the interior either, since there must be some contents inside (otherwise

is not 3-connected!). ↯

(b) Again w.l.o.g. assume C bounds the outer face. Then all of $G \setminus C$ are in the interior of C. Since we assume $G \setminus C$ is disconnected, we retrieve a component A, and the rest are given the name B (not necessarily a component). Then



(Note that when we erase B, A is not harmed because they are separated)



↯ ∎

Theorem 20 (Whitney)

Any 3-connected planar graph has only one possible embedding modulo isomorphism.

Proof. Suppose to the contrary that two embeddings $\mathcal{H}$ and $\mathcal{H}'$ of a 3-connected planar graph G are non-isomorphic. By definition of isomorphism, ~~[illegible]~~ there should be an abstract face boundary in $\mathcal{H}$ that doesn't bound any face in $\mathcal{H}'$. But since G is 3-connected, Lemma 16 tells us that any face boundary is a cycle. Hence, we have:

$$\exists \text{ Cycle } C: \; C \text{ bounds a face in } \mathcal{H} \text{ but not so in } \mathcal{H}'.$$

Then we apply Lemma 19 and deduce: C has a chord or $G \setminus C$ is separated.

(a) C has a chord.

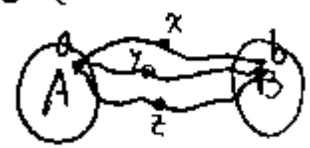
Look at the embedding $\mathcal{H}$. Without loss of generality assume C bounds the outer face. Then all stuffs are in the interior of C.

Again, by our old trick, the chord breaks C into A and B, and due to 3-connectivity there should be a path, disjoint of C, that connects A and B. Clearly it will cross the chord and leads to contradiction. ↯

(If you really want to be formal, you could add a virtual vertex outside and connect it to all guys on C, etc.)

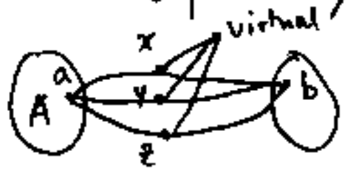
(b) C is chordless but $G \setminus C$ is separated.

Assume the separated parts are A and B (which is nonempty). ~~[illegible]~~ By 3-connectivity, we could find three disjoint paths



from $a \in A$ to $b \in B$. All of them intersect with C (for C cuts A&B ~~[illegible]~~ apart). Denote the intersections x, y, z (which are distinct).

As in (a), we look at $\mathcal{H}$, and all the stuffs are inside C. So we could again create a virtual vertex outside and connect it with x, y, z without detriment to planarity. We obtain



and unfortunately, a $K_{3,3}$ subdivision is on the spot. ■