# Planarity Test

Corollary 9 is a necessary condition for planar graphs, but it is not sufficient. For instance, we know that $K_{3,3}$ is nonplanar, yet it satisfies $|E| \leq 3|V| - 6$.

Even the stronger bound $|E| \leq 2|V| - 4$ we derived from the exersice for bipartite graphs is not sufficient. Consider the bipartite graph:

It's non-planar since it contains $K_{3,3}$. But it still satisfies $|E| \leq 2|V| - 4$.

So our natural quest would be: Is there a criterion that tells if a graph is planar or not? This is what we meant by the section title, "planarity test".

A classical and elegant criterion is given by Kuratowski in 1930s.

**Theorem 11** (Kuratowski)

$G$ is planar $\Leftrightarrow$ $G$ contains no subdivisions of $K_5$ or $K_{3,3}$.

This is a TONCAS theorem: the "$\Rightarrow$" part is obvious, for otherwise we would be able to embed $K_5$ or $K_{3,3}$ in the plane. On the other hand, the "$\Leftarrow$" part is highly non-trivial, and we shall give a proof later.

For the moment, let's walk through the rest of story. There is a very similar criteria for planarity test:

**Theorem 12** (Wagner)

$G$ is planar $\Leftrightarrow$ $G$ contains neither $K_5$ nor $K_{3,3}$ as minors.

It is known that for any fixed graph $H$, we could decide in $O(n^2)$ time whether $H$ is a minor of $G$. Hence the above theorem gives rise to an $O(n^2)$ algorithm to test planarity.

Theorem 12 is actually a special case of a very deep and magnificient result in modern graph theory, the Graph Minor Theorem:

**Theorem 13** (Robertson & Seymour)

Any class of graphs closed under the minor operation can be characterized by a finite set of forbidden minors.

The class of planer graphs is clearly closed under the minor operation, and the "forbidden set" could be chosen as $\{K_5, K_{3,3}\}$. The beauty of the theorem is, for many other minor-closed classes $\mathcal{C}$, people don't actually know what the "forbidden set" looks like, but the theorem asserts that there exists such a finite set and consequently, testing $G \overset{?}{\in} \mathcal{C}$ could be done by some but unknown polynomial time algorithm!

We surely don't have enough space to prove the theorem here, so let us move our focus to the more moderate proof of Kuratowski's Theorem.

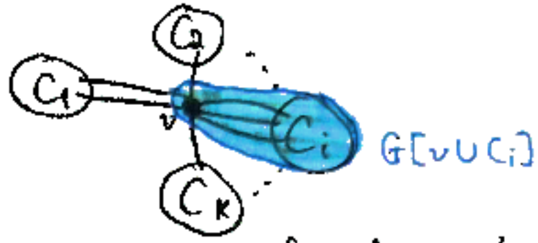
## Proof of "$\Leftarrow$" in Kuratowski's Theorem.

Suppose there's a graph $G$ that is
① nonplanar; and
② free of $K_5$ and $K_{3,3}$ as subdivision.

We take the minimal $G$ that satisfies properties ①② and try to derive a contradiction.

### Step 1: $G$ is biconnected.

If vertex $v$ is a cut vertex that separates the remainder of $G$ into components $C_1, \dots, C_k$, then at least one of the components, say $C_i$, has $G[v \cup C_i]$ nonplanar. Otherwise, all of $G[v \cup C_i]$ are planar and we could draw them individually on a plane. Using Theorem 10 we could always draw $v$ on the outer boundary. Then we "glue" these individual drawings at $v$ to produce a plane embedding of $G$, which is impossible.




Now that we found a ~~sub~~ proper subgraph $G[v \cup C_i] \subseteq G$ that satisfies ①②, we reach a contradition that $G$ is minimal. Hence $G$ cannot contain a ~~vert~~ cut vertex (i.e. is biconnected).


### Step 2: $\delta(G) \geq 3$

If $\delta(G) = 2$ then we could find some $v$: $\deg(v) = 2$. Let $N(v) =: \{u, w\}$.

(a) If $uw \in E$, then $G - v$ is nonplanar. Otherwise we could draw it on plane so that $uw$ lies on the oter boundary. Adding $uv$ and $wv$ back will give us an embedding of $G$, a contradiction to nonplanarity.

Hence $G-v$ satisfies ①②. but this again contradicts the minimality of $G$.

(b) If $uw \notin E$, then by minimality we know $G - \{uv, vw\} + uw$ is planar or contains $K_5/K_{3,3}$ as subdivision.


It can't be planar though (for the same reason as in (a)). So it contains a $K_5/K_{3,3}$ as subdivision then. If the subdivision doesn't use edge $uw$ at all, then it is also contained in $G$, a contradiction. If the subdivision uses $uw$, we could break it into a longer path $uvw$, so it is also contained in $G$, a contradiction.

In regard of (a)(b), we always end up contradicting $\delta(G)=2$. Hence $\delta(G) \geq 3$ (It can't be $\delta(G) < 2$ since $G$ is biconnected)

## Step 3: Find $H \subsetneq G$ biconnected and planar.

This step is a direct consequence of Steps 1 & 2. Recall from graph theory that a biconnected graph could always be decomposed to an initial cycle plus some appended "ears":

we shall use it later; call it $e = xy$.

Consider the final "ear" that we added. It must be an edge rather than a path longer than 2! (For otherwise the interior of the path would have degree $2 \leq \delta(G) = 3$) But this simply tells us that if we remove this final edge, the subgraph $H \subsetneq G$ is still biconnected. By the minimality of $G$, we see that $H$ is either planar or contains $K_{3,3}/K_5$ as subdivision, but the latter is impossible. So $H$ is planar.


$G = H + e$

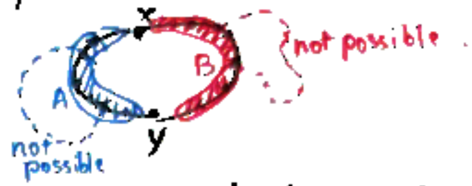
## Step 4: Analyse $H$ and $G$.

Since $H$ is biconnected, we could always find a cycle (in $H$) that goes through $x$ and $y$. And, since $H$ is planar, there is always a way to draw it on the plane.

Now we go over all possible embeddings of $H$ and all possible cycles in $H$ through $x$ & $y$, and choose the (embedding, cycle) combination that maximises the # faces inside the cycle (denoted $C$). We shall analyse in detail the picture of $C$ and its interior/exterior.

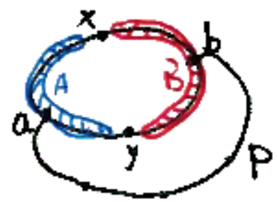

Vertices $x$ and $y$ naturally split $C$ into two non-empty segments, $A$ and $B$.

One observation would be: if $a, a' \in A$ connected via a path, then the path could not lie in the exterior of $C$, otherwise the maximality of $C$ is violated. Similarly, if $b, b' \in B$ are connected via a path then the path should ~~also~~ never be in the exterior of $C$.

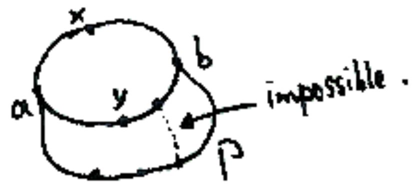



The second observation is: there must be something lying in the exterior of $C$. Otherwise, we could route $x \to y$ outside $C$, giving an embedding for $G$!

So what to put there in the exterior of $C$? There should be a $A$-$B$ path $P$; we denote the endpoints $a \in A$ and $b \in B$.



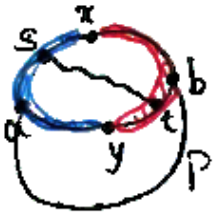

(This doesn't violate the maximality since the cycle $x$-$b$-$a$-$x$ doesn't contain $y$) Also note that none of the internal vertices in $P$ could have a "shortcut" to the cycle $C$; otherwise the maximality of $C$ is violated.



Enough for the exterior. Now turn our eyes on the interior of $C$. There must be something inside as well; in particular, the "something" must split the interior so that $x$ and $y$ ~~[illegible]~~ bounds different faces.

We enumerate all possibilities:

(a)

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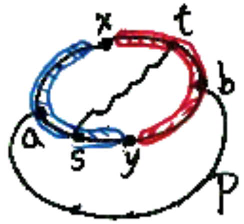


⇓

In $G$ we could find a $K_{3,3}$


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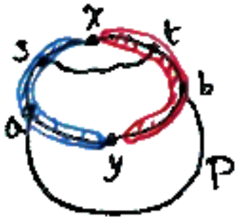
(b)

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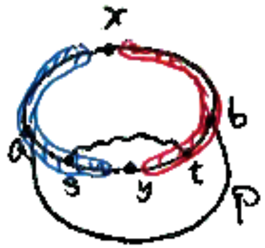
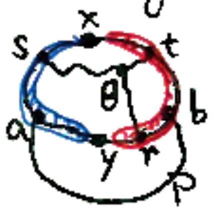



Take the left picture for instance. The problem is: if there's nothing else, then we could "flip" the whole ~~exterior~~ component that contains $P$ in $G \setminus C$ to interior of $C$ (how? → Exercise) without crossing. This would add new faces into the interior, contradicting maximality.

To prevent cheating, we have to add something else. The only possibility is

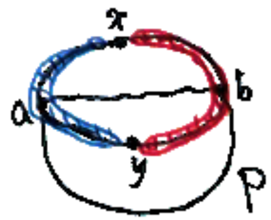



~~where θ could be the [illegible] s or t, but r [illegible] be strictly between [illegible]~~

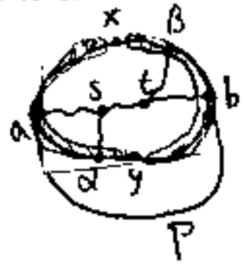
Note that $\theta \neq s, t$ (otherwise we could flip $\theta r$ out and $P$ in. Also, $r \neq a, b$. Hence the path $s-\theta-r$ serves the same purpose as the path in (a), and we could find a $K_{3,3}$ subdivision in $G$.

[There's actually one nuance: what if $r=y$? Try to find a subdivision of $K_{3,3}$!]

(c)




As in (b), we need more structures in to avoid $P$ flipping in. But this time we must take care of both upper and lower halves!




Here s and t are strictly between a and b but we allow s = t. Also, α, β ≠ a, b and could coincide with y, x, respectively.

We leave the verification to reader that G always contains a subdivision of $K_{3,3}$ or $K_5$ in this case.

To conclude our long proof, we find out that G in any situation would contain subdivision of $K_{3,3}$ / $K_5$, a direct contradiction to assumption ②. Therefore, ①② are inconsistent and, if we insist that ① is true, then ② must be false. ■

Remark. One more reminder about step 1: we claimed ~~that~~ during the proof that we could "draw v on the outer boundary". The argument was: select an edge e incident to v and a face f bounded by e, then flip f to the outer face, so v automatically goes out.

We have seen through Theorems 11 - 13 various planarity tests, but all of them are quite theoretical than algorithmic. There are in fact a large pool of algorithmic results on planarity testing, some of which even runs in linear time. Moreover, a few of them no only <u>decides</u> if a graph is planar but also <u>constructs</u> an embedding (not strictly in the sense we defined, though) when one exists. These results are too involved to state here. For the curious, refer to DMP algorithm ($O(n^2)$) or Hopcroft-Tarjan algorithm ($O(n)$).