More Topology

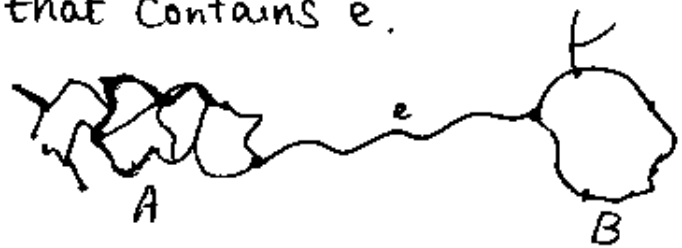
This section presents some important topological results without proofs. Arguably, they are "obvious" that most people could take them for granted easily. Based on these results, we assemble a few useful lemmas for later use.

Theorem 4 (Jordan curve theorem)
A Jordan curve $C \subseteq \mathbb{R}^2$ would partition $\mathbb{R}^2$ into exactly two regions. That is, $\mathbb{R}^2 \setminus C$ has two regions. Both of them have C as their boundaries.

Theorem 5 ("Inverse" Jordan curve theorem)
For disjoint sets $A, B \subseteq \mathbb{R}^2$, each a union of finitely many arcs and points, and for arc e that connects A and B with $\mathring{e} \cap A = \mathring{e} \cap B = \emptyset$, we have:

$R \setminus e$ is a region of $\mathbb{R} \setminus (A \cup B \cup e)$

where R is the region of $\mathbb{R} \setminus (A \cup B)$ that contains e.



Remark. Simply put, R is the region that $A \cup B$ "carves", pretending that e doesn't exist. The theorem states that, even if we added e back, $R \setminus e$ is still a whole piece, i.e. not disconnected by e. The intuition is clear: we could always walk a long way around from one side of e to the other side.
We call this "inverse Jordan curve theorem" because it shows that "a non-Jordan curve (i.e. an arc) could not separate a plane into two regions".

~~Exercise. Show that any face~~

Lemma 6

Let G be a plane graph and $H \subseteq G$ be a subgraph. Then $\forall f \in F(G)$, $\exists f' \in F(\mathcal{H}) : f \subseteq f'$.

Moreover, if $\partial f \subseteq \mathcal{H}$ then $f = f'$.

proof. The first part is almost trivial: removing vertex/edges could never disconnect more points. Formally, $\forall x, y \in f$, by definition there's an arc $\subseteq f \subseteq \mathbb{R}^2 \setminus G \subseteq \mathbb{R}^2 \setminus H$ that connects x and y, so x and y lie in the same face of $\mathcal{H}$. Hence $f \subseteq f'$.

To prove the second part, assume for contradiction that $f \subsetneq f'$. Take

$x \in f$ and $y \in f' \setminus f$. Then by a previous exercise, any arc connecting x and y would intersect with $\partial f \subseteq \mathcal{H}$, so in fact there's no way to connect x and y in $\mathbb{R}^2 \setminus H$, contradicting that $x, y \in f'$. ■

Lemma 7

Let G be a plane graph.
$\forall e \in E(G)$, $\forall f \in F(G)$, we have

(1) either $e \subseteq \partial f$ or $\mathring{e} \cap \partial f = \emptyset$; that is, ∂f is a union of multiple <u>whole</u> edges.

(2) if e lies on a cycle C (of the abstract graph) then e is <u>on exactly 2</u> faces, say f_1 and f_2, the boundary of and they belong to different regions separated by C.

(3) if e doesn't lie on a cycle, then it is on the boundary of exactly 1 face.

proof.

Our proof strategy is as follows:

<u>Step 1.</u> Fix a point $x \in \mathring{e}$. Show that x lies on the boundary of exactly 2 or 1 faces, depending on whether e is on a cycle.

Step 2 For any other point $y \in \mathring{e}$, we show that y is incident to exactly the same face as x does.

Step 3 Hence, the entire $\mathring{e}$ would either have 2 or 1 incident face(s), depending on whether e is on a cycle.

Step 4 But the conclusion could be extended to the endpoints as well.

Here we use the "stronger" definition

Now we do step 1. in Proposition 2, i.e. every arc should be finitely linear. Such assumption would make our lives easier.

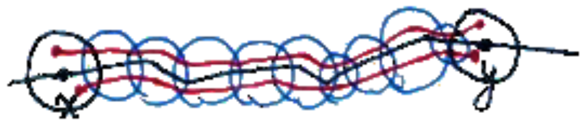
Then, we choose $r > 0$ small enough so that $U(x_e, r)$ intersects with at most 2 segments. $U(x_e, r)$ is then cut into 2 half discs.

Let f and f' be the faces that they reside in. (f might $= f'$)

If e lies on a cycle, then by Theorem 4 the cycle separates the plane $\mathbb{R}^2$, and e (thus x_e) is on the boundary of both regions. From Lemma 6 we know that adding other edges/vertices besides the cycle will not connect these regions. So f and f' have to be distinct.

Following a similar argument, but this time using Theorem 5 in place of 4, we could show that $f = f'$ if e doesn't lie on a cycle.

Next we perform step 2. This is a routine application of Theorem 1: Find a finite disc cover along x to y, then it's always possible to draw an arc "close enough" to e that connects the corresponding half discs of x and y.



Therefore, the "upper half discs" of x and y are equivalent, and so are the "lower half-discs". So x and y must have identical incident faces, proving (2) (3).

As an exercise, please show (1) using (2)(3). ■

We have seen just now two rigorous proofs of some "obvious" common sense. Apart from providing a solid basis of our intuition, the more important point is really to acquaint the reader with the methodology that people formalise daily intuitions.

EULER'S FORMULA

Euler's formula is perhaps the first "amazing" result so far:

Theorem 8 (Euler's Formula)
For any connected planar graph $G = (V, E)$ and any embedding of it, we have

$$|V| - |E| + |F| = 2$$

where $|F|$ is the number of faces in the embedding.

Proof. We fix V and incrementally add edges to the graph until we reach the full edge set E.

Base case: $|V| - 1$ edges

Then the graph is a tree. By repeatedly applying Theorem 5 one could show that only 1 face is present in any embedding of a tree. So

$$|V| - |E| + |F| = |V| - (|V| - 1) + 1 = 2.$$

Incremental step

Every time we add an edge e, it lies on some cycle C. From Lemma 7 we know that $e \subseteq \partial f_1, \partial f_2$ for exactly 2 faces f_1 and f_2. What we will show next is again something "obvious": e breaks an original face apart into 2 faces f_1 and f_2.

Let G^- be the graph before we add e.

Since $G^- \subseteq G$, by Lemma 6, $\exists f \in F(G^-)$: $f_1 \subseteq f$ and also $f' \in F(G^-)$: $f_2 \subseteq f'$.

But $f = f'$, since in G^- we could start from any point from f, get arbitrarily close to $\dot{e}$ (because $f_1 \subseteq f$ and $e \subseteq \partial f_1$), "cross" the position of e, and enter the realm of f'. Such scheme connects f and f', implying $f = f'$. Therefore, $e \cup f_1 \cup f_2 \subseteq f$.

In what follows we check that e does not harm any other faces. Again, since $G^- \subseteq G$, by Lemma 6 we know every $f_0 \in F(G)$, $f_0 \neq f_1, f_2$, is contained in some $f_0^* \in F(G^-)$.

But $\dot{e} \cap \partial f_0 = \emptyset$ as $f_0 \neq f_1, f_2$, by Lemma 7

so $\partial f_0 \subseteq G^-$ and, from the "moreover" part of Lemma 6, $f_0 = f_0^*$.

~~This proves that every face except f~~

Finally, we observe that

$$\underbrace{\left(\bigcup_{\substack{f_0 \in F(G)\\ f_0 \neq f_1, f_2}} f_0\right)}_{\text{these shown to be "}\subseteq\text{"}} \cup \underbrace{(e \cup f_1 \cup f_2)}_{} = \underbrace{\left(\bigcup_{\substack{f_0^* \in F(G^-)\\ f_0^* \neq f}} f_0^*\right)} \cup f$$

they were also shown to be "$\subseteq$"

And all the parts are disjoint. So the "$\subseteq$" must be "$=$".

In conclusion, we showed that

$F(G) = (F(G^-) \setminus \{f\}) \cup \{f_1, f_2\}$

or $|F(G)| = |F(G^-)| + 1$

So $|V| - |E| + |F|$ is preserved. ■

Corollary 9
Any planar graph $G=(V,E)$ satisfies
$|E| \le 3|V| - 6$

Proof. Fix an embedding of G. Let
$T := \{(e,f) \in E \times F : e \subseteq \partial f\}$

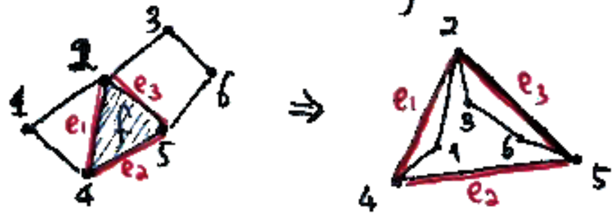
- For each fixed e, the # of $f \in F$ s.t. $e \subseteq \partial f$ is no larger than 2, thus $|T| \le 2|E|$.
- For each fixed f, the # of $e \in E$ s.t. $e \subseteq \partial f$ is at least 3 (why?), thus $|T| \ge 3|F|$
- Combined: $2|E| \ge 3|F|$
- plugging this into Euler's Formula and substituting $|F|$ gives the result. ■

Remark.
(1) One could also derive $|F| \le 2|V| - 4$.
(2) It shows that planar graphs are sparse.
(3) The proof uses "double counting", a recurring and powerful technique in Combinatorics. We did it explicitly by constructing T; but typically people prefer skipping the construction and argue right away.

Exercise
(1) Show that K_5 is non-planar.
(2) Modify the proof for bipartite planar graph to get a better bound. Then use it to prove that $K_{3,3}$ is non-planar.

(We end this section by a smart result not really related to Euler's Formula.

Theorem 10
Let $G=(V,E)$ be a planar graph, and suppose that edges $e_1, \cdots, e_k$ bounds a face in some embedding of G. Then there is an embedding of G where $e_1, \cdots, e_k$ bounds the outer face.



Proof.

There's a continuous bijection g between $\mathbb{R}^2$ and

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x^2+y^2+(z-1)^2 = 1 \\ z \neq 2 \end{array} \right\},$$

defined as follows:

$g(x,y)$ is the intersection of the line (x,y)—$(0,0,2)$ and the sphere S. Since g is a bijection and is continuous, all the desired structures are preserved when mapping plane graph G to $g(G)$ — the properties include the edge/vertex/face relation, e.g. e is on the boundary of $f \iff g(e)$ is on the boundary of $g(f)$.

After mapping G to $g(G)$, we rotate the S so that the "north pole" $(0,0,2)$ is contained in the region bounded by $g(e_1), \dots, g(e_k)$. Then we map the resulting image back to $\mathbb{R}^2$, which ensures that $e_1, \dots, e_k$ ~~lies on~~ bound the outer face. ∎