Topology Basics

This course is about geometry, and a large bulk of the study is about drawing visual elements in some space (e.g. $\mathbb{R}^2$, $\mathbb{R}^3$, the surface of a sphere). As computer scientists, we usually care about a <u>finite</u> system of visual elements. How do the elements interact? What are the combinatorial properties of the system? And how can we efficiently construct, store, manipulate and traverse the system algorithmically? These are typical questions that we try to tackle.

The first part of the notes concerns with drawing an abstract graph in the plane $\mathbb{R}^2$ — in the "typical" way where a point represents a vertex and an arc (curve) represents an edge. But what is an arc exactly and how do we argue about it? Of course one may resort to his intuition, but it turns out that all these intuitive concepts could be made formal and rigorous by a deep field called topology. We will touch some basics of topology and axiomize some results from topology, in this and the following section. The point is to convince the readers of the possibility of formalisation if they wish.

def. open & closed.

Let $A \subseteq S$ be a set where S is the "ambient space" or "universe". A is called <u>open</u> if $\forall a \in A, \exists r > 0$: the neighbourhood $U(a,r) \subseteq A$. It is called <u>closed</u> if $S \setminus A$ is open; equivalently, if the limit of any convergent sequence in A is contained in A, too. (why?)

Remark. In the definition we used the notion of "neighbourhood" $U(a,r)$, which relies upon the notion of distance. So S is implicitly assumed a metric space.

But we remind the reader that, in pure topological sense, there are other definitions of open/closed sets which don't require distance at all! Such defs are too general for our purpose.

def. **compact set**: both bounded and closed.

Theorem 1. A set $A \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ For any family $\mathcal{F} = \{F_i\}$, $F_i \subseteq \mathbb{R}^n$ open, and $\bigcup_{F \in \mathcal{F}} F \supseteq A$, "cover of A" we could find a finite subfamily $\mathcal{F}' \subseteq \mathcal{F} : \bigcup_{F \in \mathcal{F}'} F \supseteq A$.

Now we are prepared for the more "geometry" side of topology.

def. **arc and Jordan curve.**

Let $f : [0,1] \to \mathbb{R}^n$ be a continuous function.

(1) If f is injective, then we call its image an arc. $f(0)$ and $f(1)$ are the endpoints.

(2) If $f(0) = f(1)$ and f is injective on $[0,1)$, then we call its image a Jordan curve.

(We shall denote $\mathring{e} := e \setminus$ endpoints of e for arc e.)

~~The definition is really natural. We don't want to deal with discontinuous~~

(Just a side note: the definition could be rephrased in concise topological terms "an arc is homeomorphic to $[0,1]$; a Jordan curve is homeomorphic to a unit circle".)

We call $a, a' \in A \subseteq \mathbb{R}^n$ connected (denoted $a \overset{A}{\sim} a'$) if $\exists$ an arc $\subseteq A$ whose endpoints are a and a'. Clearly, $\overset{A}{\sim}$ is an equivalence relation, thus classifying A into one or more equivalence classes. We call each of these classes a region. If A is open, then so are its regions. (why?)

def boundary.

The boundary of region R, denoted ∂R, is all the points $x \in \mathbb{R}^n$ s.t.

$$\forall r>0,\ U(x,r)\cap R \neq \emptyset \ \wedge\ U(x,r)\cap(\mathbb{R}^n\setminus R) \neq \emptyset.$$

i.e. lies exactly upon the "separation" of R and $\mathbb{R}^n \setminus R$.

At this point, we don't have much tools except some basic definitions and a well-known theorem. But these are sufficient for the core definitions of this course: plane and planar graphs.

Exercise. Show that if the endpoints of an arc e lie in different regions, say R_1 and R_2, then $e\cap\partial R_1 \neq \emptyset$, $e \cap \partial R_2 \neq \emptyset$.

PLANARITY

For the moment, we shall make a little twist of the concept of graphs:

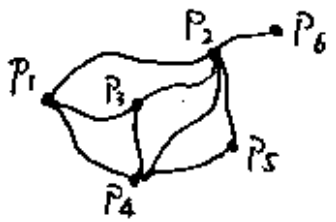
def plane graph.

a plane graph is a pair (V,E) where

(1) $V \subseteq \mathbb{R}^2$ is a finite set of points.

(2) $E \subseteq \mathbb{R}^2$ is a finite set of arcs whose endpoints are always in V. Also, we require that $\forall e,e' \in E$, $\underbrace{\mathring{e}\cap\mathring{e}'=\emptyset}_{\text{no crossing}}$ and $\underbrace{|e\cap e'|\leq 1}_{\text{no multi-edge}}$.

So strictly speaking, a plane graph is not a graph, because they are different types of objects. However if we inspect the above definition carefully, we see that there's always an "abstract" graph behind the scenes. Hence we don't actually distinguish a plane graph or its corresponding abstract graph.



a plane graph

its corresponding abstract graph: $(\{P_1, \cdots, P_6\}, \{P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, \cdots\})$

Though a plane graph does correspond to an abstract graph, the converse is not necessarily true (and we shall prove that sections later). The reason is that a plane graph imposes the strong restriction that no two edges could cross in the plane, while a general graph doesn't care about ~~that~~ the ~~spec~~ concrete graphical representation.

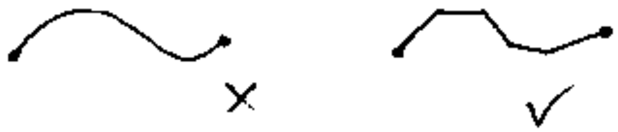
def. planar graph.

An abstract graph G is planar if we could find a plane graph G' that G corresponds to G'.

If G is planar, then we call the G' above an <u>embedding</u> of G.
not unique.

Proposition 2.

In the definition of plane graph, we could impose an even stronger condition without changing the meaning:
"all the arcs are required to be ~~a~~ finite consecutive line segments".



proof. The intuitive idea is to cut the arcs (edges) fine enough and straighten piecewise so that no edges would cross. To make the argument rigorous, we appeal to Theorem 1.

For each edge e and $x \in \mathring{e}$, define $d(x)$ to be the minimum distance between x and all other edges. Clearly $d(x) > 0$. Now we draw a square with side length $d(x)$, centered at x. So we collect a family of squares $\mathcal{F}$ after going through all $x \in \mathring{e}$.

Obviously $\mathcal{F}$ covers $\mathring{e}$, so by Theorem 1 we could find a finite subfamily $\mathcal{F}'$ that covers $\mathring{e}$ as well.

Since by our choice of squares, $\left(\bigcup_{F\in\mathcal{F}'} F\right) \cap e' = \emptyset$ for all $e' \neq e$, it's always safe to walk along the sides of these squares without crossing other edges. This gives us the desired "finite linearization" of edge e. ■

def. face.

For a plane graph $G=(V,E)$, the set $\mathbb{R}^2\setminus G$ (which means $\mathbb{R}^2\setminus(V\cup E)$) is open (why?), whose regions are called the faces of G. We use $F(G)$ to denote the set of faces of G.

G

$\mathbb{R}^2\setminus G$

Proposition 3

A plane graph G always has one unbounded face (called "outer face"). All other faces (if any) are bounded.

proof. Since G is finite, and any arc is bounded (why?), there exists a "bounding box" that contains the entire G. The exterior of the box clearly belongs to the same equivalence class in $\mathbb{R}^2\setminus G$, so $\mathbb{R}^2\setminus G$ has at least one unbounded region, ~~hence~~ i.e. G has at least one unbounded face. As two faces must be disjoint, the rest of the faces must reside in the bounding box, thus bounded. ■